2. Probability

Probability is the quantification of the likelihood that an event will occur on a scale between 0 and 1. Probability is widely used in decision making or future prediction by calculating probabilities for various situations.

2.1 Probability

☞ Think	When a coin is tossed, a head or a tail appears. If a product is manufactured in one factory and inspected, it is either good or defective.
Exploration	What are common characteristics such as the above examples that appear frequently in our daily life?

- Tossing a coin, inspecting a product, and similar things are often repeated around us. We know that the possible consequences of these things are either 'head' or 'tail', 'good' or 'defective', but we don't know what the outcome will be. An experiment or observation in which similar things are repeated and all possible outcomes are known, but one outcome is determined by chance is called a trial.
- The set of all possible outcomes that can occur in a trial is called a **sample space**, and a subset of this sample space is called an **event**. An event made up of one element is called a **basic event**.
- The sample space is usually denoted by S and events are denoted by uppercase English letters, A, B, C, ... , etc. For example, in a coin tossing trial, when ‘head’ is H and ‘tail’ is T, the sample space is as follows:

$$S = \{H,\ T\}$$

Events of ‘head’ and ‘tail’ are denoted as follows:

$$A = \{H\},\ B = \{T\}$$

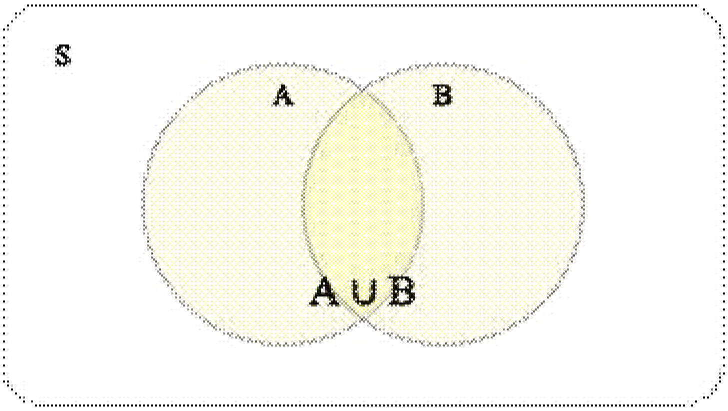
Example 2.1	When you roll a single dice, find the followings. 1) Sample space S 2) Event A with odd numbers
Solution	1) The sample space for rolling a single dice is as follows: $S = \{1, 2, 3, 4, 5, 6\}$ 2) The event with odd numbers is as follows: $A = \{1, 3, 5\}$

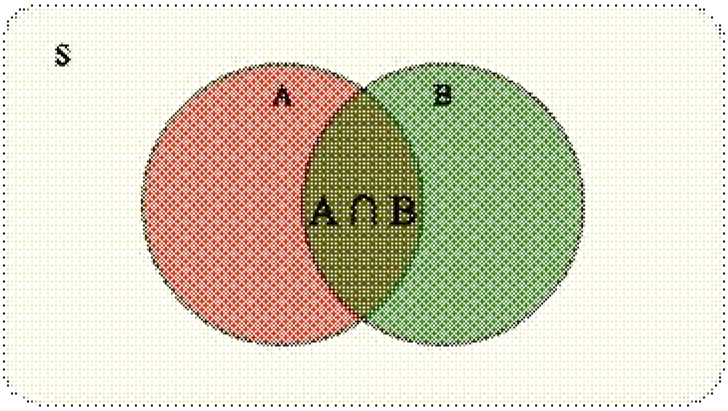
Example 2.2	When one product is inspected, the result is marked as 'good' (O) or 'defective' (X). If you inspect two products, find the followings. 1) Sample space S 2) Event A with one 'good' and one 'defective'
Solution	1) The sample space for inspecting two products is as follows: S = {OO, OT, TO, TT} 2) The event A with one 'good' and one 'defective' is as follows: A = {OT, TO}

Exercise 2.1	When a coin is tossed, let H be the event that the coin comes up 'head', and T be the event that it comes up 'tail'. If you toss a coin twice, find the followings. 1) Sample space S 2) Event A in which 'head' appears once

Exercise 2.2	When a dice is rolled twice, find the followings. 1) Sample space S 2) Event A in which the same number of dots appears twice 3) Event B in which the sum of the dots appeared in each roll is less than 3 or equal to 3.

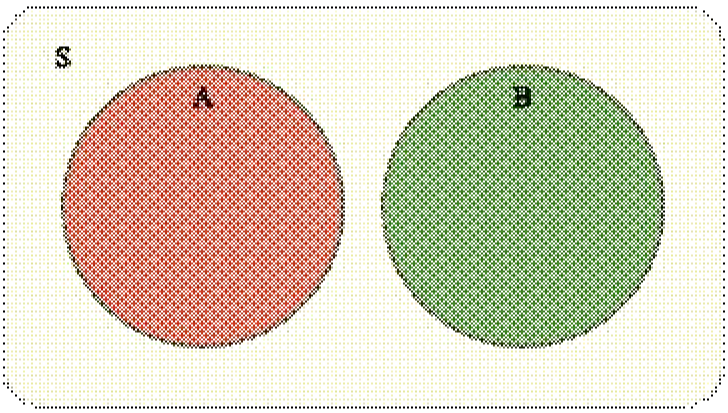
- With respect to two events A and B in the sample space S, the event in which A or B occurs is denoted by $A \cup B$, and the event in which A and B occurs simultaneously denoted by $A \cap B$.





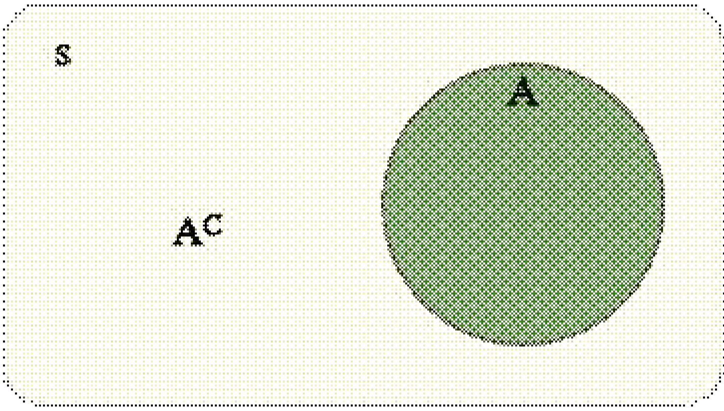
<Figure 2.1> Event $A \cup B$ and event $A \cap B$

- If event A and B do not occur simultaneously which is $A \cap B = \varnothing$, two events are called **mutually exclusive.**



<Figure 2.2> Mutually exclusive events A and B

- When there is an event A, the event in which A does not occur is called a **complementary event** of A and denoted by A^C. Since A and A^C cannot occur at the same time, that is $A \cap A^C = \varnothing$, the events A and A^C are mutually exclusive.



<Figure 2.3> Event A and complementary event A^C

Example 2.3	When a dice is rolled, an event that has an odd number of dots is called A and an event that has the number of dots which is greater than or equal to 3 is called B. Find the following events. 1) $A \cup B$ 2) $A \cap B$ 3) A^C
Solution	The sample space by rolling a dice is S = {1, 2, 3, 4, 5, 6}, event A is {1, 3, 5}, and event B is {1, 2, 3}. 1) $A \cup B$ = {1, 2, 3, 5} 2) $A \cap B$ = {1, 3} 3) A^C = {2, 4, 6}

Exercise 2.3	When a coin is tossed twice, A is an event in which 'head' appears more than once and B is an event in which 'tail' appears more than once. Are the two events mutually exclusive?

A. Statistical Probability

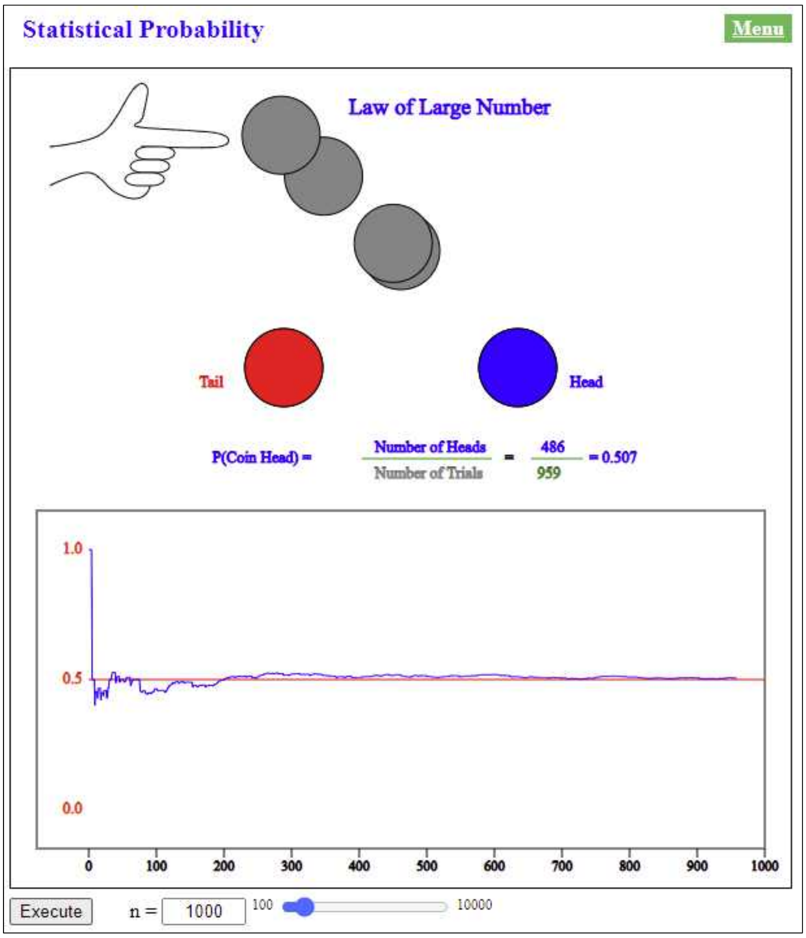
☞ Think	If you toss a coin, the likelihood of getting head (H) or tail (T) seems to be half and half. Probability is a numerical representation of this likelihood. Is it correct to define the probability of getting head as a half?
Exploration	Let's toss a coin 10, 50, or 100 times to find the number of heads and check their relative frequencies of head whether it is a half or not.

- An example of finding the relative frequency by actually tossing a coin 10, 50, or 100 times and writing down the number of heads is as follows:

Number of tosses (n)	10	50	100
Number of heads (x)	4	23	51
Relative frequency (x/n)	0.40	0.46	0.51

When the number of tosses is small, the relative frequency of the number of heads may not be 0.5. However, as the number of tosses of a coin increases, the relative frequency approaches 0.5. If you experiment more with the number of tossing a coin using a computer, it can be seen that the relative frequency gradually approaches 0.5 as shown in <Figure 2.4>. This relative frequency is called the **statistical probability** of an event that a coin comes up head.

Example 2.4	Let's experiment a coin tossing simulation using 『eStatH』.
Solution	If you select 'Statistical Probability' from the 『eStatH』 menu, a window as shown in <Figure 2.4> appears. If you enter n = 10000 and click [Execute] button, the coin tossing will be executed 10,000 times, and you can observe the relative frequency of the number of heads of the coin.



<Figure 2.4> Simulation experiment for tossing a coin in 『eStatH』

Exercise 2.4	If a dice is made perfectly, the probability of getting each side when you roll the dice is likely $\frac{1}{6}$. Roll the dice 12, 36, or 120 times to see if it is $\frac{1}{6}$ really.

- In general, the probability of an event A, denoted as $P(A)$, is expressed as a numerical value between 0 and 1. Let n_A be the number of times an event A occurs when the trial is repeated n times. As the number of trials increases sufficiently, if the relative frequency $\frac{n_A}{n}$ approaches a constant value p, this value is called the **statistical probability** of the event A which is denoted as $P(A) = \frac{n_A}{n}$.

☞ Statistical probability

Let n_A be the number of times an event A occurs when a trial is repeated n times. As the number of trials increases sufficiently, if the relative frequency $\frac{n_A}{n}$ approaches a constant value p, this value is called the **statistical probability** of the event A which is denoted as $P(A) = \frac{n_A}{n}$.

Example 2.5	The demographic structure of Korea by age in 2020 is as follows: Age | Population (unit 1000) | Relative Frequency 0-14 | 6,300 | 0.124 15-64 | 37,360 | 0.736 65 and over | 7,070 | 0.139 Find the probability that one randomly chosen Korean is over 65 years of age.
Solution	In 2020, there are 7.07 million Koreans aged 65 and over, and the relative frequency is 0.139. In other words, the statistical probability of meeting a person over 65 in Korea is 0.139.

Age	Population (unit 1000)	Relative Frequency
0-14	6,300	0.124
15-64	37,360	0.736
65 and over	7,070	0.139

Exercise 2.5	Youths aged between 13 and 18 was interviewed whether they are physically healthy or not in 2017 and the result is as follows: Find the probability that one randomly selected youth aged between 13 and 18 is healthy.

‘Are you healthy?’	Relative frequency
not al all healthy	0.001
not so healthy	0.021
healthy	0.496
very healthy	0.482

- It is not easy to define the probabilities of all events using statistical probabilities and use them for decision-making in reality. In order to apply the probability easily to the real problem, a mathematical probability based on the model was studied.

B. Mathematical Probability

☞ Think	A dice is made in the shape of a cube, with one to six dots drawn on each side.
Exploration	Since a dice was made in the shape of a cube, the probability of each side when the dice was rolled would be equal to 1/6. There are many games that use this dice, but are these games fair?

- In order to obtain the statistical probability of each side of the dice, numerous trials of rolling the dice are required. The statistical probability obtained through sufficient trials will be close to 1/6. However, it is not easy in reality to do many trials to find the probability of one event.
- If one dice is rolled, the sample space is {1, 2, 3, 4, 5, 6}. Assume that each element in the sample space has the same likelihood of occurrence. The probability of each element can be said to be 1/6. A mathematical probability is defined as the likelihood of an event under the rational assumption that each element in the sample space is equally likely to appear. When making decisions about real problems, this mathematical probability is used a lot.
- In general, assuming that each element of the sample space S in a trial has the same probability of occurrence, the mathematical probability that an event A will occur denoted as $P(A)$ is defined as follows:

$$P(A) = \frac{n(A)}{n(S)}$$

$n(S)$ is the number of elements in the sample space
$n(A)$ is the number of elements in event A

In general, the statistical probability p of an event A approaches to the mathematical probability if the number of trials is large enough.

☞ Mathematical probability

Assume that each element of the sample space S in a trial has the same probability of occurrence. The mathematical probability that an event A will occur denoted as $P(A)$ is defined as follows:

$$P(A) = \frac{n(A)}{n(S)}$$

$n(S)$ is the number of elements in the sample space
$n(A)$ is the number of elements in event A

Example 2.6	An office worker went on a business trip to a city, and there were two restaurants (Restaurant 1 and Restaurant 2) near his accommodation. He hesitated about which restaurant to go. He rolled a dice and counted the number of dots that appear on the top. If an odd number came up, he decided to go to Restaurant 1, and if it was even, he decided to go to Restaurant 2. What is the probability that Restaurant 1 would be chosen?
Solution	The sample space of rolling a dice trial which counts the number of dots appearing on the top is S = {1, 2, 3, 4, 5, 6}. The event of odd numbers is A = {1, 3, 5}. The number of elements in the sample space is $n(S)$= 6 and the number of elements in the event A is $n(A)$ = 3. Therefore, the probability of Restaurant 1 being selected is as follows: $$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = 0.5$$

Example 2.7	In a production plant which produces a product, you want to take out 3 products from a box containing 10 products at the same time and inspect the products. If there are 8 good products and 2 defective products in this box, find the probability of getting 2 good products and 1 defective product.

Example 2.7 Solution	The number of all cases which take out 3 out of 10 products is as follows: $_{10}C_3 = 120$ If S is the sample space, $n(S) = 120$. If the event in which there are 2 good products and 1 defective product is A, then $n(A) = {}_8C_2 \times {}_2C_1 = 56$ Hence, the probability of the event A is as follows: $P(A) = \frac{n(A)}{n(S)} = \frac{56}{120}$

Exercise 2.6	Let the event A be that an odd number of dots is appeared in a single dice roll. Find the mathematical probability that the event A will occur.

Exercise 2.7	When two different dice are rolled at the same time, find the mathematical probability that the sum of the dots appeared on the top of each dice is less than 5.

C. Basic Properties of Probability

☞ Think	Can an event have a probability more than 1? Or could it be less than zero?
Exploration	Probability is a number from 0 to 1 indicating the likelihood that an event will occur.

- Let's examine properties of the mathematical probability when each element in the sample space has the same probability of occurrence. Since an event A is a subset of the sample space S, $n(S)$ should be greater than or equal to $n(A)$ which should be greater than 0 as follows:

$$0 \le n(A) \le n(S)$$

If we divide each term by $n(S)$, it becomes as follows:

$$0 \le \frac{n(A)}{n(S)} \le 1$$
$$0 \le P(A) \le 1$$

- If A is equal to the sample space S, then $P(S) = \frac{n(S)}{n(S)} = 1$.
- If A is an empty set ϕ, then $P(\phi) = \frac{n(\phi)}{n(S)} = 0$.

☞ Basic properties of probability

Assume that each element in the sample space has the same probability of occurrence and A is an event in the sample space S.

1) $0 \leq P(A) \leq 1$

2) $P(S) = 1$

3) $P(\phi) = 0$

Example 2.8	You want to place four people A, B, C, and D on four chairs next to each other. Find the total number of cases in which four people are placed and the number of cases in which A is placed on the leftmost. What is the probability that A is placed to the leftmost?
Solution	The number of elements in the sample space in this problem is as follows: (Number of people who can be placed on the far left) × (Number of people who can be placed in the second position, excluding the left) × (Number of people who can be placed in the third position except for the two on the left) × (Number of people who can be placed on the right except for the three on the left) = 4 × 3 × 2 × 1 = 4! = 24 The event in which A is placed on the leftmost is the number of placing the remaining 3 people are in the second, third and right positions except for A, so 3×2×1 = 3!. Therefore, the probability that A is placed on the leftmost is as follows: 3! / 4! = 6/24 = 0.25

Exercise 2.8	A company has 4 security guards (A, B, C, D). Every morning, two of these guards are randomly selected and one of them is assigned to the front door and the other to be at the back door. Find the total number of cases where 4 people are placed at the front and rear doors, and the number of cases where A is placed at the front door. What is the probability that A will be placed at the front door?

2.2 Addition Rule of Probability

☞ Think	A survey of 50 middle school students found that 30 students watched the soccer game between Korea and Japan during the last Olympics on TV and 20 students watched the volleyball game. There were 10 students who watched both the soccer and volleyball game.
Exploration	What is the probability of a student who watched either the soccer or volleyball?

- If the event where a student watched the soccer game is A and the volleyball is B, the event of watching either the soccer or volleyball is $A \cup B$. Denote the number of element in each event of A, B and $A \cup B$ as $n(A)$, $n(B)$ and $n(A \cap B)$ respectively, then they satisfy the following relation.

  $$n(A \cup B) \;=\; n(A) + n(B) - n(A \cap B)$$

  If we divide both sides by $n(S)$, it becomes as follows:

  $$\frac{n(A \cup B)}{n(S)} \;=\; \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

  Therefore, the probability that either the event A or event B occurs, denoted as $\mathrm{P}(A \cup B)$, is as follows:

  $$\mathrm{P(A \cup B) \;=\; P(A) + P(B) - P(A \cap B)}$$

  This is called an **addition rule of probability**. In the above example, it becomes as follows:

  $$\mathrm{P(A \cup B)} \;=\; \frac{30}{50} + \frac{20}{50} - \frac{10}{50} \;=\; \frac{40}{50}$$

- If two events A and B are mutually exclusive, then $\mathrm{P}(A \cap B)$ = 0 and the addition rule becomes as follows:

$$P(A \cup B) = P(A) + P(B)$$

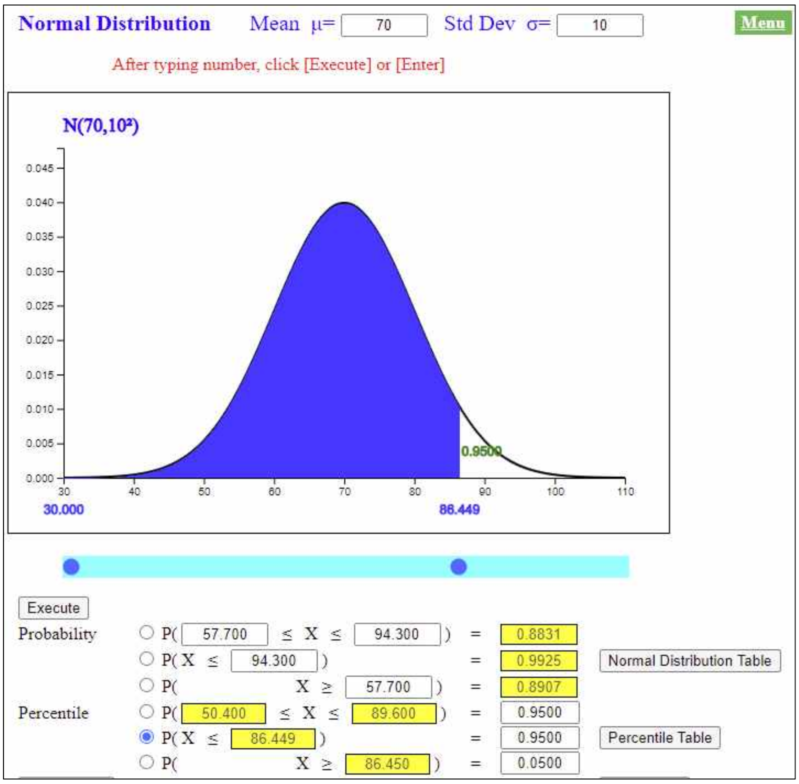
☞ Addition rule of probability

The probability of either the event A or event B occurs, $P(A \cup B)$, is as follows:

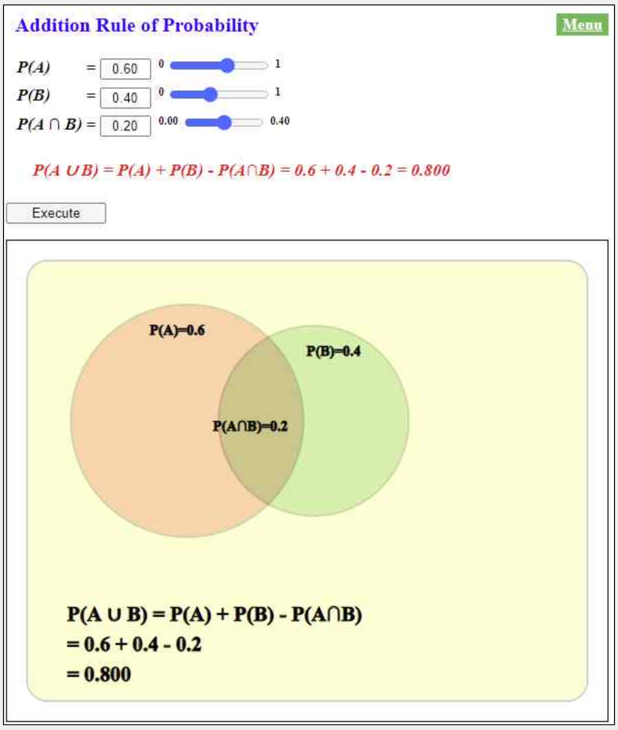
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If two events A and B are mutually exclusive, then $P(A \cap B) = 0$ and the addition rule becomes as follows:

$$P(A \cup B) = P(A) + P(B)$$



<Figure 3.32> Calculation of percentile in normal distribution

Example 2.9	Let us practice the addition rule of probability in the above example using 『eStatH』.
Solution	If you select 'Addition Rule of Probability' from the 『eStatH』 menu, a window such as in <Figure 2.5> appears. Enter $P(A)$ = 0.6, $P(B)$ = 0.4, $P(A \cap B)$ = 0.2 and click [Execute] button to observe the graph of the addition rule of probability. We can examine the addition rule by changing $P(A)$, $P(B)$, and $P(A \cap B)$ using the slide bars.  <Figure 2.5> Addition rule of probability

Exercise 2.9	Among 80 first-year students of business administration at a university, 50 students took economics, 30 students took political science, and 20 students took both courses. If you meet a first-year business administration student, what is the probability that this student will be taking either economics or political science?

A. Probability of Complementary Event

☞ Think	The probability of getting the number 1 or 2 when a dice is rolled is $\frac{2}{6}$.
Exploration	What is the probability of getting remaining numbers 3, 4, 5, 6?

- The sample space of rolling a dice is S = {1, 2, 3, 4, 5, 6}. If the event in which the number of dots appeared is 1 or 2 is A = {1, 2}, the event of remaining numbers 3, 4, 5, 6 is called a **complementary event** and is denoted as A^C = {3, 4, 5, 6}. Since the event A and the complementary event A^C are mutually exclusive, the addition rule becomes as follows:

$$P(A \cup A^C) = P(A) + P(A^C)$$

Since $P(A \cup A^C) = P(S) = 1$, the probability of the complementary event $P(A^C)$ becomes as follows:

$$P(A^C) = 1 - P(A)$$

> ☞ Probability of the complementary event
>
> If A^C is the complementary event of A, the probability of A^C is as follows:
>
> P(A^C) = 1 − P(A)

- Probability calculation of the complementary event often contains the words 'at least' or 'remainder'.

Example 2.10	Consider a box with 6 products, and 2 of them are defective. What is the probability that at least one defective item is found when three items are sampled for product inspection? Assume that the product once sampled for inspection is not put back in again (without replacement).
Solution	The probability that one defective product was found in 3 product inspections was (${}_4C_2$ × ${}_2C_1$) / ${}_6C_3$ = 3/5. lso, the probability of finding two defective products is (${}_4C_1$ × ${}_2C_2$) / ${}_6C_3$= 4/20 = 1/5. Therefore, the probability of finding at least one defective product is 3/5 + 1/5 = 4/5. Another way to find this probability is to find the probability of an event in which there will be no defective items (this is called the complementary event of an event in which at least one defective item is found) and subtract it from 1. That is, the probability that at least one defective product is found is as follows: 1 − (${}_4C_3$ / ${}_6C_3$) = 1 − (4/20) = 4/5

Exercise 2.10	When a family of 6 including the parents eat at the round table, what is the probability that the parents are not adjacent?

2.3 Conditional Probability and Multiplication Rule of Probability

☞ Think	Out of 40 third-year high school students, 24 were male students and 16 were female students. 8 male students and 4 female students wear glasses.
Exploration	1) What is the probability of being a male student when choosing one of these high school students randomly? 2) When a student is randomly selected, what is the probability that this student is a male student and wears glasses? 3) If one of the male students was randomly selected, what is the probability that this student wore glasses?

- The number of events given in this problem can be summarized in the following table by denoting the event of male student as A, the event of female student as F, the event of wearing glasses as B, and the event of not wearing glasses as N.

	Wearing Glasses(B)	Not Wearing Glasses(N)	Total
Male(A) Female(F)	$n(A \cap B)$ = 8 $n(F \cap B)$ = 4	____ ____	$n(A)$ = 24 $n(F)$ = 16
Total	____	____	$n(S)$ = 40

- The above table makes it easy to calculate the number of events not given by the problem as follows:

	Wearing Glasses(B)	Not Wearing Glasses(N)	Total
Male(A) Female(F)	$n(A \cap B)$ = 8 $n(F \cap B)$ = 4	$n(A \cap N)$ = 16 $n(F \cap N)$ = 12	$n(A)$ = 24 $n(F)$ = 16
Total	$n(B)$ = 12	$n(N)$ = 28	$n(S)$ = 40

- 1) If you randomly select one of the high school students, the probability of being a male student is as follows:

$$P(A) = \frac{n(A)}{n(S)} = \frac{24}{40}$$

2) If you randomly select a student, the probability that this student is a male student and wears glasses is as follows:

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{8}{40}$$

3) When one of the male students is randomly selected, the probability that this student wears glasses is denoted as a symbol $P(B|A)$, and, since the case of a male student M can be viewed as a sample space, it is calculated as follows:

$$P(B|A) = \frac{n(A \cap B)}{n(A)} = \frac{8}{24}$$

$P(B|A)$ is called the **conditional probability** of wearing glasses (B) when a male student is selected (A)

- If S is a sample space of a statistical experiment, the conditional probability of the event B when the event A occured is as follows:

$$P(B|A) = \frac{n(A \cap B)}{n(A)}$$

If we divide both numerator and denominator by $n(S)$, the conditional probability becomes as follows:

$$P(B|A) = \frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{P(A\cap B)}{P(A)}$$

☞ Conditional probability

The conditional probability of the event B when the event A occurred is as follows:

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \quad (단\ P(A)\neq 0)$$

- .In the table above, the probabilities of two events occurring at the same time are summarized as follows: This is called a **joint probability table**.

	Wearing Glasses(B)	Not Wearing Glasses(N)	Total
Male(A)	$P(A\cap B) = \frac{8}{40}$	$P(A\cap N) = \frac{16}{40}$	$P(A) = \frac{24}{40}$
Female(F)	$P(F\cap B) = \frac{4}{40}$	$P(F\cap N) = \frac{12}{40}$	$P(F) = \frac{16}{40}$
Total	$P(B) = \frac{12}{40}$	$P(N) = \frac{28}{40}$	$P(S) = 1$

- In this table, the probability $P(A\cap B) = \frac{8}{40}$ can be calculated by multiplying the probability of male students $P(A) = \frac{24}{40}$ with the conditional probability $P(B|A) = \frac{8}{24}$.

$$P(A\cap B) = P(A)\times P(B|A) = \frac{24}{40}\times\frac{8}{24} = \frac{8}{40}$$

It is called a **multiplication rule of probability**. This formula can be derived if you multiply $P(A)$ on both side of the definition of the conditional probability $P(B|A) = \frac{P(A\cap B)}{P(A)}$.

$$P(A\cap B) = P(A)\times P(B|A)$$

- The probability $P(A \cap B) = \frac{8}{40}$ can be calculated by multiplying the probability of wearing glasses $P(B) = \frac{12}{40}$ with the conditional probability $P(A|B) = \frac{8}{12}$.

$$P(A \cap B) = P(B) \times P(A|B) = \frac{12}{40} \times \frac{8}{12} = \frac{8}{40}$$

- In general, the multiplication rule of probability is as follows:

> ☞ Multiplication rule of probability
>
> The multiplication rule of two events A and B is as follows:
>
> $$P(A \cap B) = P(A) \times P(B|A) \quad (P(A) \neq 0)$$
>
> $$P(A \cap B) = P(B) \times P(A|B) \quad (P(B) \neq 0)$$

- Suppose that the joint probability table on gender of students and wearing glasses is as follows:

	Wearing Glasses(B)	**Not Wearing Glasses(N)**	**Total**
Male(A)	$P(A \cap B)$ = **12/40**	$P(A \cap N)$ = **12/40**	$P(A)$ = **24/40**
Female(F)	$P(F \cap B)$ = **8/40**	$P(F \cap N)$ = **8/40**	$P(F)$ = **16/40**
Total	$P(B)$ = **20/40**	$P(N)$ = **20/40**	$P(S)$ = **40/40**

Here, the probability of all students wearing glasses $P(B) = \frac{20}{40} = \frac{1}{2}$ is the same as that of male students $P(B|A) = \frac{12}{24} = \frac{1}{2}$. If the occurrence of an event A does not affect the probability that the event B will occur, that is,

$$P(B|A) = P(B)$$

it is said that two events A and B are **independent** of each other. On the other hand, if two events A and B are not independent, it is said two events are **dependent**.

- If two events A and B are independent, the rule of multiplication becomes as follows:

$$P(A \cap B) = P(A) \times P(B|A) = P(A) \times P(B)$$

Inversely, if $P(A \cap B) = P(A) \times P(B)$ and $P(A) \neq 0$, the conditional probability becomes as follows:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

Therefore, two events A and B are independent.

☞ **Multiplication rule of probability when two events are independent**

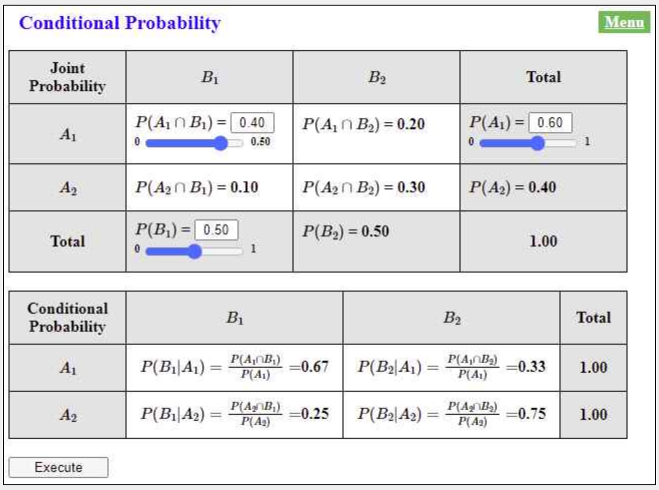
A necessary and sufficient condition that two events A and B are independent is as follows:

$$P(A \cap B) = P(A) \times P(B) \quad (\text{단}, \; P(A) \neq 0, \; P(B) \neq 0)$$

- In the table above, all events for the joint probability are independent of each other. In this case, it is said that the gender variable and the variable of wearing glasses are independent. Looking at the table, the probabilities of "wearing glasses" and "not wearing glasses" events of all students are 0.5 and 0.5, respectively. When the two variables are independent, these ratios are maintained for each male and female student.

Example 2.11	Let's practice the conditional probability and independent events using 『eStatH』.
Solution	When 'Conditional Probability' is selected from the 『eStatH』 menu, a window such as <Figure 2.6> appears. Here, if the joint probability is adjusted, the conditional probability can be observed below it, and a line graph such as <Figure 2.7> and a bar graph of the conditional probability for each row can be observed.

Example 2.11 Solution (continued)

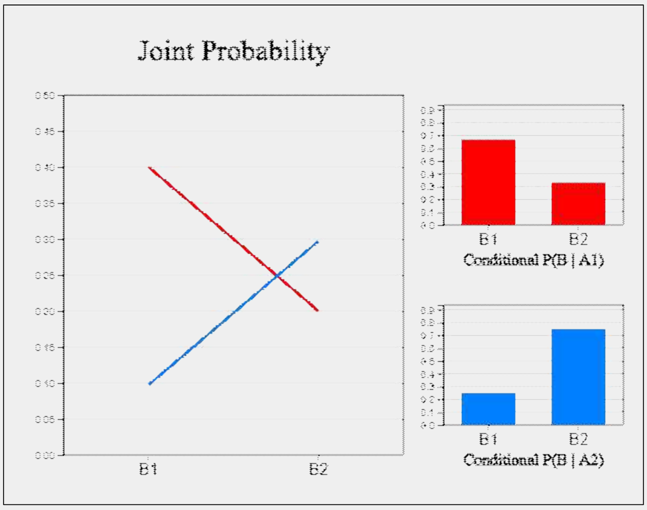

Conditional Probability

Joint Probability	B_1	B_2	Total
A_1	$P(A_1 \cap B_1) =$ 0.40	$P(A_1 \cap B_2) = 0.20$	$P(A_1) =$ 0.60
A_2	$P(A_2 \cap B_1) = 0.10$	$P(A_2 \cap B_2) = 0.30$	$P(A_2) = 0.40$
Total	$P(B_1) =$ 0.50	$P(B_2) = 0.50$	1.00

Conditional Probability	B_1	B_2	Total
A_1	$P(B_1 \mid A_1) = \frac{P(A_1 \cap B_1)}{P(A_1)} = 0.67$	$P(B_2 \mid A_1) = \frac{P(A_1 \cap B_2)}{P(A_1)} = 0.33$	1.00
A_2	$P(B_1 \mid A_2) = \frac{P(A_2 \cap B_1)}{P(A_2)} = 0.25$	$P(B_2 \mid A_2) = \frac{P(A_2 \cap B_2)}{P(A_2)} = 0.75$	1.00

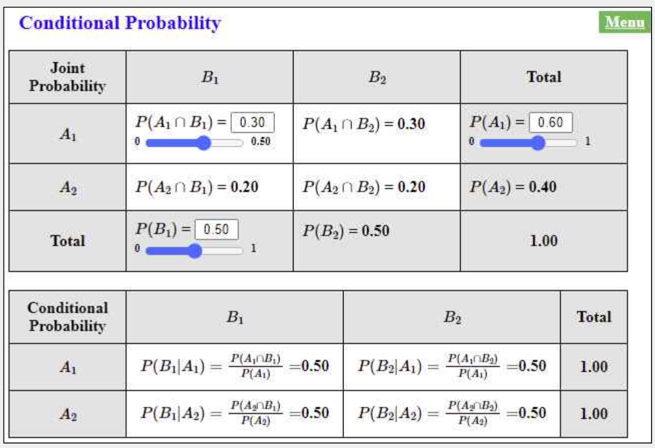
Execute

<Figure 2.6> Conditional probability



<Figure 2.7> Graph of joint probabilities and their conditional probability

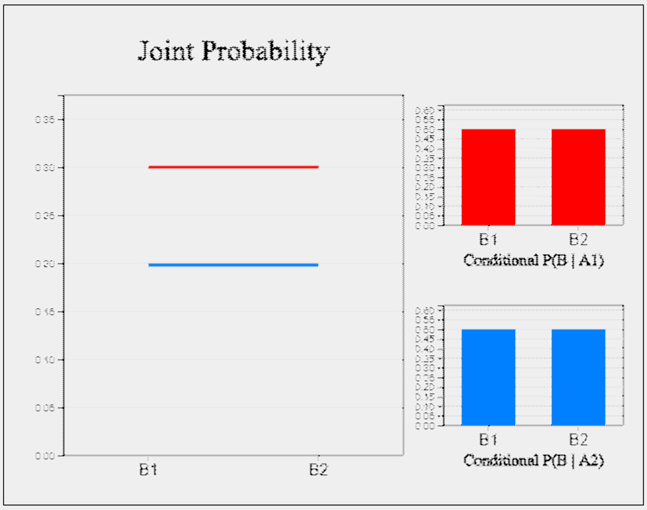
If you adjust the joint probability as $P(A_1 \cap B_1)$ = 0.30, $P(A_1)$ = 0.60 becomes as in <Figure 2.8>. <Figure 2.9> displays a line graph and conditional probability in case of independent events.


Conditional Probability

Joint Probability	B_1	B_2	Total
A_1	$P(A_1 \cap B_1) =$ 0.30	$P(A_1 \cap B_2) = 0.30$	$P(A_1) =$ 0.60
A_2	$P(A_2 \cap B_1) = 0.20$	$P(A_2 \cap B_2) = 0.20$	$P(A_2) = 0.40$
Total	$P(B_1) =$ 0.50	$P(B_2) = 0.50$	1.00

Conditional Probability	B_1	B_2	Total
A_1	$P(B_1 \mid A_1) = \frac{P(A_1 \cap B_1)}{P(A_1)} = 0.50$	$P(B_2 \mid A_1) = \frac{P(A_1 \cap B_2)}{P(A_1)} = 0.50$	1.00
A_2	$P(B_1 \mid A_2) = \frac{P(A_2 \cap B_1)}{P(A_2)} = 0.50$	$P(B_2 \mid A_2) = \frac{P(A_2 \cap B_2)}{P(A_2)} = 0.50$	1.00

<Figure 2.8> Probabilities when two variables are independent

Example 2.11 Solution (continued)	 <Figure 2.9> Line graph and conditional probabilities when two variables are independent.

Exercise 2.11	Let A be the event that the number of dots by drawing a dice is odd and B be the event that the number of dots is less than 3. Examine whether the two events are independent or dependent.

Exercise 2.12

The distribution of 30 freshmen at a university by gender and region is as follows:

	Seoul (S)	Country (C)	Total
Male(A)	10	10	20
Female(F)	5	5	10
Total	15	15	30

When a student is selected, is the case of Male and Seoul independent of each other?

Exercise

2.1 If two events A and B are mutually exclusive, what is the probability of the event $A \cup B$? (Answer ③)

① P(A)－P(B) ② P(A)P(B)

③ P(A)＋P(B) ④ P(A)/P(B)

2.2 Let the probability of the event A be P(A) and of the event B be P(B). Which of the followings is not right? (Answer ②)

① $0 \le P(A) \le 1$ ② $-1 \le P(B) \le 0$

③ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ④ $P(A \cap B) = P(A) \cdot P(B|A)$

2.3 If P(A)=0.4, P(B)=0.2 and P(A | B)=0.6, what is $P(A \cap B)$? (Answer ②)

① 0.08 ② 0.12

③ 0.24 ④ 0.48

2.4 When $A \subset B$, compare the conditional probability P(A | B) with P(A)? (Answer ①)

① the same or large ② small

③ the same or small ④ cannot be compared

2.5 When two dices are rolled simultaneously, what is the probability that 2 dots and 5 dots appear simultaneously? (Answer ④)

① $\frac{1}{3}$ ② $\frac{1}{6}$

③ $\frac{1}{12}$ ④ $\frac{1}{18}$

2.6 When we roll a dice three times, what is the probability that the first, the second, and the third will come out with 5 dots, 3 dots and even number of dots? (Answer ②)

① $\frac{1}{30}$ ② $\frac{1}{72}$

③ $\frac{1}{108}$ ④ $\frac{1}{276}$

2.7 If you randomly select three bulbs one by one without replacement from a container containing five good bulbs and two defective bulbs, find the probability that one of them is defective? (Answer ④)

① $\frac{1}{7}$ ② $\frac{2}{7}$

③ $\frac{3}{7}$ ④ $\frac{4}{7}$

2.8 If P(A)=0.4, P(B)=0.2 and P(A | B)=0.6, what is P(A∩B)? (Answer ②)
① 0.08 ② 0.12
③ 0.24 ④ 0.48

2.9 If P(A)=0.4, P(B)=0.2 and P(A | B)=0.6, what is P(B | A)? (Answer ③)
① 0.08 ② 0.24
③ 0.30 ④ 0.40

2.10 Denote the number of dots when we roll two dices as (x_1, x_2) and consider an event $B=\{(x_1,x_2) | x_1>x_2\}$. What is the probability of P(B)? (Answer ③)
① $\frac{1}{3}$ ② $\frac{1}{12}$
③ $\frac{5}{12}$ ④ $\frac{1}{36}$

2.11 Denote the number of dots when we roll two dices as (x_1, x_2) and consider two events $A=\{(x_1, x_2) \mid x_1+x_2=10\}$, B=\{(x_1, x_2) | $x_1 > x_2$\}. What is the probability P(B | A)? (Answer ①)
① $\frac{1}{3}$ ② $\frac{1}{12}$
③ $\frac{5}{12}$ ④ $\frac{1}{2}$

Answer
2.1 ③, 2.2 ②, 2.3 ②, 2.4 ①, 2.5 ④, 2.6 ②, 2.7 ④, 2.8 ②, 2.9 ③, 2.10 ③, 2.11 ①