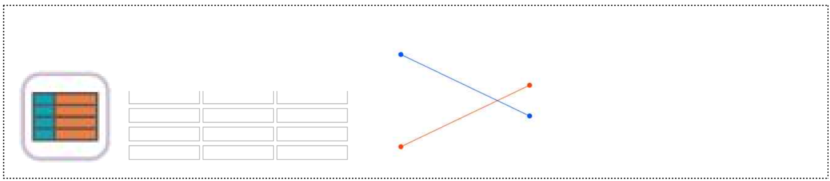
11

Testing Hypothesis for Categorical Data



SECTIONS

CHAPTER OBJECTIVES

The hypothesis tests that we have studied from Chapter 7 to Chapter 10 are for continuous data. In this chapter, we describe testing hypothesis for categorical data.

Section 11.1 describes the goodness of fit test for the frequency table of categorical data.

Section 11.2 describes the independence and homogeneity tests for the contingence table of two categorical data.

11.1 Goodness of Fit Test

- The frequency table of categorical data discussed in Chapter 4 counts the frequency of possible values of a categorical variable. If this frequency table is for sample data from a population, we are curious what would be the frequency distribution of the population. The goodness of fit test is a test on the hypothesis that the population follows a particular distribution based on the sample frequency distribution. In this section, we discuss the goodness of fit test for categorical distributions (Section 11.1.1) and the goodness of fit test for continuous distribution (Section 11.1.2).

11.1.1 Goodness of Fit Test for Categorical Data

- Consider the goodness of fit test for a categorical distribution using the example below.

Example 11.1.1

The result of a survey of 150 people before a local election to find out the approval ratings of three candidates is as follows. Looking at this frequency table alone, it seems that A candidate has a 40 percent approval rating, higher than the other candidates. Based on this sample survey, perform the goodness of fit test whether three candidates have the same approval rating or not. Use 『eStatU』 with the 5% significance level.

Candidate	Number of Supporters	Percent
A	60	40.0%
B	50	33.3%
C	40	25.7%
Total	150	100%

Answer

- Assume each of candidate A, B, and C's approval rating is $p_1,\ p_2,\ p_3$ respectively. The hypothesis for this problem is as follows:

  H_0 : The three candidates have the same approval rating. (i.e., $p_1 = p_2 = p_3 = \frac{1}{3}$)
  H_1 : The three candidates have different approval ratings.

- If the null hypothesis H_0 is true that the three candidates have the same approval rating, each candidate will have $50 (= 150 \times \frac{1}{3})$ supporters out of total 150 people. It is referred to as the 'expected frequency' of each candidate when H_0 is true. For each candidate, the number of observed supporters in the sample is called the 'observed frequency'. If H_0 is true, the observed and expected number of supporters can be summarized as the following table.

Candidate	Observed frequency (denoted as O_i)	Expected frequency (denoted as E_i)
A	O_1 = 60	E_1 = 50
B	O_2 = 50	E_2 = 50
C	O_3 = 40	E_3 = 50
Total	150	150

Example 11.1.1 Answer (continued)

- If H_0 is true, the observed frequency (O_i) and the expected frequency (E_i) will coincide. Therefore, in order to test the hypothesis, a statistic which uses the squared difference between O_i and E_i is used. Specifically, the statistic to test the hypotheses is as follows:

$$\chi^2_{obs} = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \frac{(O_3 - E_3)^2}{E_3}$$

If the observed value of this test statistic is close to zero, it can be considered that H_0 is true, because O_i is close to E_i. If the observed value is large, H_0 will be rejected. The question is, 'How large value of the test statistic would be considered as the statistically significant one?' It can be shown that this test statistic approximately follows the chi-square distribution with $k-1$ degrees of freedom if the expected frequency is large enough. Here k is the number of categories (i.e., candidates) in the table and it is 3 in this example. Therefore, the decision rule to test the hypotheses is as follows:

'If $\chi^2_{obs} > \chi^2_{k-1\,;\,\alpha}$, reject H_0, else do not reject H_0'

- The statistic χ^2_{obs} can be calculated as follows:

$$\chi^2_{obs} = \frac{(60-50)^2}{50} + \frac{(50-50)^2}{50} + \frac{(40-50)^2}{50} = 4$$

Since the significance level α is 5%, the critical value can be found from the chi-square distribution as follows:

$$\chi^2_{k-1\,;\,\alpha} = \chi^2_{3-1\,;\,0.05} = \chi^2_{2\,;\,0.05} = 5.991$$

Therefore, H_0 cannot be rejected. In other words, although the above sample frequency table shows that the approval ratings of the three candidates differ, this difference does not provide sufficient evidence to conclude that the three candidates have different approval ratings.

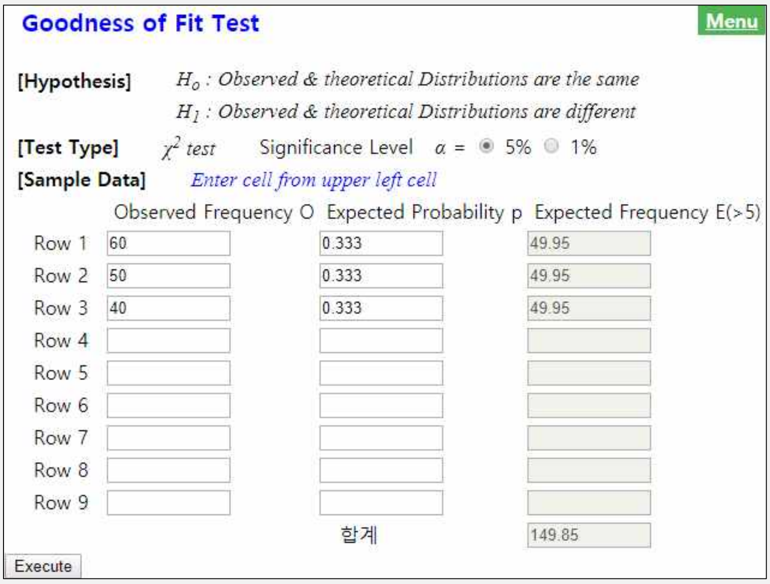
- Since sample approval ratings of each candidate are $\hat{p}_1 = \frac{60}{150} = 0.40$, $\hat{p}_2 = \frac{50}{150} = 0.33$, $\hat{p}_3 = \frac{40}{150} = 0.27$, the 95% confidence intervals for the population proportion of each candidate's approval rating using the formula ($\hat{p} \pm 1.96\sqrt{\hat{p}(1-\hat{p})/n}$) (refer Chapter 6.4) are as follows:

$$\text{A} : 0.40 \pm 1.96\sqrt{\frac{0.40 \cdot 0.60}{150}} \quad \Leftrightarrow \quad [0.322, 0.478]$$
$$\text{B} : 0.33 \pm 1.96\sqrt{\frac{0.33 \cdot 0.67}{150}} \quad \Leftrightarrow \quad [0.255, 0.405]$$
$$\text{C} : 0.27 \pm 1.96\sqrt{\frac{0.27 \cdot 0.73}{150}} \quad \Leftrightarrow \quad [0.190, 0.330]$$

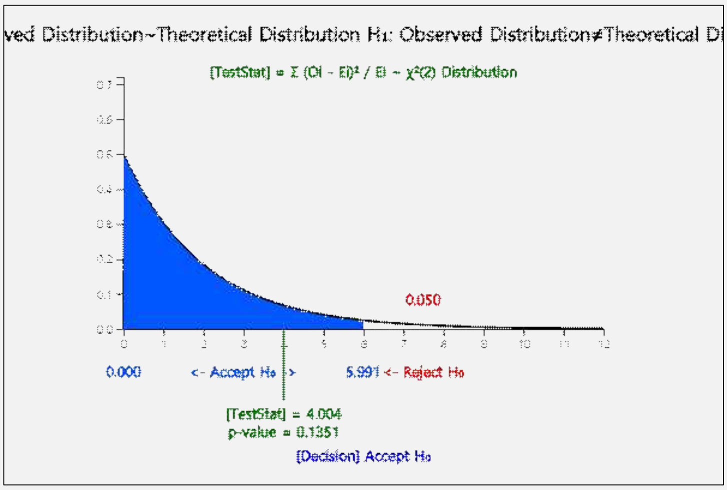
The overlapping of the confidence intervals on the three candidates' approval ratings means that one candidate's approval rating is not completely different from the other.

- In the data input box that appears by selecting the 'Goodness of Fit Test' of 『eStatU』, enter the 'Observed Frequency' and 'Expected Probability' data as shown in <Figure 11.1.1>. After entering the data, select the significance level and click [Execute] button to calculate the 'Expected Frequency' and to see the result of the chi-square test. Be sure that this chi-square goodness of fit test should be applied when the expected frequency of each category is at least 5.

Example 11.1.1 Answer (continued)



<Figure 11.1.1> Goodness of fit test in 『eStatU』



<Figure 11.1.2> 『eStatU』 Chi-square Goodness of Fit Test

- Consider a categorical variable X which has k number of possible values $x_1, x_2, \cdots, x_k$ and their probabilities are $p_1, p_2, \cdots, p_k$ respectively. In other words, the probability distribution for the categorical variable X is as follows:

X	$x_1 \quad x_2 \quad \cdots \quad x_k$	Total
$P(X=x)$	$p_1 \quad p_2 \quad \cdots \quad p_k$	1

- When random samples are collected from the population of the categorical random variable X and their observed frequencies are $(O_1, O_2, \cdots, O_k)$, the hypothesis to test the population probability distribution of $(p_1, p_2, \cdots, p_k) = (p_{10}, p_{20}, \cdots, p_{k0})$ is as follows:

H_0 : Distribution of $(O_1, O_2, ..., O_k)$ are from the distribution $(p_1, p_2, \cdots, p_k) = (p_{10}, p_{20}, \cdots, p_{k0})$
H_1 : Distribution of $(O_1, O_2, ..., O_k)$ are not from the distribution $(p_1, p_2, \cdots, p_k) = (p_{10}, p_{20}, \cdots, p_{k0})$

- If the total number of samples n is large enough, the above hypothesis can be tested using the chi-square test statistic as follows:

'If $\chi^2_{obs} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi^2_{k-m-1\,;\,\alpha}$, then reject H_0'

Here, $(E_1, E_2, \cdots, E_k) = (np_{10}, np_{20}, \cdots, np_{k0})$ are expected frequencies, m is the number of population parameters estimated from the sample data. In [Example 11.1.1], since there was not a population parameter estimated from the sample, $m = 0$.

Goodness of Fit Test

Consider a categorical variable X which has k number of possible values $x_1, x_2, \cdots, x_k$ and their probabilities are $p_1, p_2, \cdots, p_k$ respectively. Let observed frequencies for each value of X from n samples are $(O_1, O_2, \cdots, O_k)$, expected frequencies for each value of X from n samples are $(E_1, E_2, \cdots, E_k) = (np_{10}, np_{20}, \cdots, np_{k0})$ and the significance level is α.

Hypothesis:

H_0 : Distribution of $(O_1, O_2, ..., O_k)$ follows $(p_{10}, p_{20}, \cdots, p_{k0})$
H_1 : Distribution of $(O_1, O_2, ..., O_k)$ does not follow $(p_{10}, p_{20}, \cdots, p_{k0})$

Decision Rule:

'If $\chi^2_{obs} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi^2_{k-m-1\,;\,\alpha}$, then reject H_0'
m is the number of population parameters estimated from the samples.

☞

In order to use the chi-square Goodness of Fit test, all expected frequencies E_i should be greater than 5.

A category which has an expected frequency less than 5 can be merged with other category.

[Practice 11.1.1]

Market shares of toothpaste A, B, C and D are known to be 0.3, 0.6, 0.08, and 0.02 respectively. The result of a survey of 100 people for the toothpaste brands are as follows. Can you conclude from these data that the known market share is incorrect? Use 『eStatU』. $\alpha = 0.05$.

Brand	A	B	C	D	Total
Number of Customers	192	342	44	22	600

11.1.2 Goodness of Fit Test for Continuous Data

- The goodness of fit test for categorical data using the chi-square distribution can also be used for continuous data. The following is an example of the goodness of fit test in which data are derived from a population of a normal distribution. The parametric statistical tests from Chapter 6 to Chapter 9 require the assumption that the population is normally distributed and the goodness of fit test in this section can be used to test for normality.

Example 11.1.2

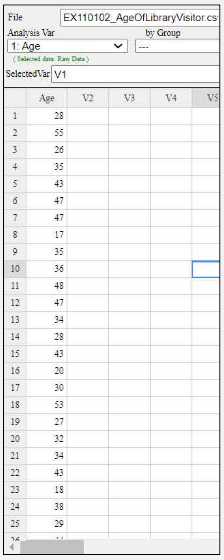
Ages of 30 people who visited a library in the morning is as follows. Test the hypothesis that the population is normally distributed at the significance level of 5%.

28 55 26 35 43 47 47 17 35 36 48 47 34 28 43
20 30 53 27 32 34 43 18 38 29 44 67 48 45 43

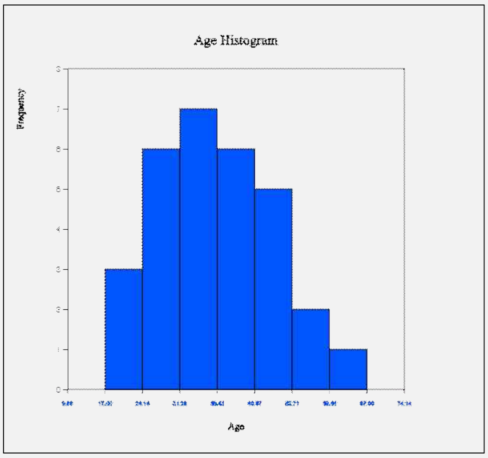
Ex ⇨ eBook ⇨ EX110102_AgeOfLibraryVisitor.csv

Answer

- Age is a continuous variable, but you can make a frequency distribution by dividing possible values into intervals as we studied in histogram of Chapter 3. It is called a categorization of the continuous data.
- Let's find a frequency table which starts at the age of 10 with the interval size of 10. The histogram of『eStat』makes this frequency table easy to obtain. If you enter the data as shown in <Figure 11.1.3>, click the histogram icon and select Age from the variable selection box, then the histogram as <Figure 11.1.4> will appear.



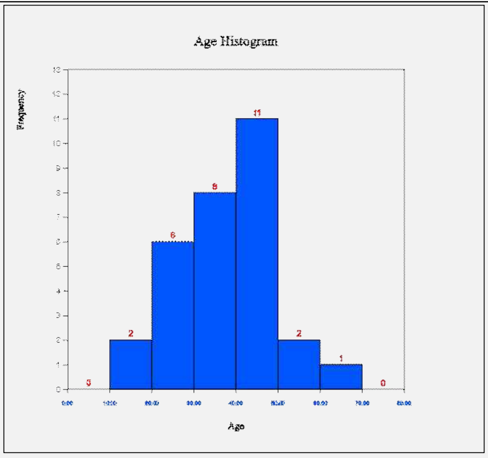
<Figure 11.1.3> Data input at『eStat』



<Figure 11.1.4> Default histogram of age

- If you specify 'start interval' as 10 and 'interval width' as 10 in the options window below the histogram, the histogram of <Figure 11.1.4> is adjusted as <Figure 11.1.5>. If you click [Frequency Table] button, the frequency table as shown in <Figure 11.1.6> will appear in the Log Area. The designation of interval size can be determined by a researcher.

Example 11.1.2 Answer (continued)



<Figure 11.1.5> Adjusted histogram of age

Histogram Frequency Table	Group Name	0
Interval (Age)		Total
1 [10.00, 20.00)	2 (6.7%)	2 (6.7%)
2 [20.00, 30.00)	6 (20.0%)	6 (20.0%)
3 [30.00, 40.00)	8 (26.7%)	8 (26.7%)
4 [40.00, 50.00)	11 (36.7%)	11 (36.7%)
5 [50.00, 60.00)	2 (6.7%)	2 (6.7%)
6 [60.00, 70.00)	1 (3.3%)	1 (3.3%)
Total	30 (100%)	30 (100%)

<Figure 11.1.6> Frequency table of the adjusted histogram

- Since the normal distribution is a continuous distribution defined at $-\infty < x < \infty$, the frequency table of <Figure 11.1.6> can be written as follows:

Table 11.1.2 Frequency table of age with adjusted interval

Interval id	Interval	Observed frequency
1	X < 20	2
2	20 $\leq$ X < 30	6
3	30 $\leq$ X < 40	8
4	40 $\leq$ X < 50	11
5	50 $\leq$ X < 60	2
6	60 $\leq$ X	1

- The frequency table of sample data as Table 11.1.2 can be used to test the goodness of fit whether the sample data follows a normal distribution using the chi-square distribution. The hypothesis of this problem is as follows:

H_0 : Sample data follow a normal distribution
H_1 : Sample data do not follow a normal distribution

- This hypothesis does not specify what a normal distribution is and therefore, the population mean μ and the population variance σ^2 should be estimated from sample data. Pressing the 'Basic Statistics' icon on the main menu of 『eStat』 will display a table of basic statistics in the Log Area, as shown in <Figure 11.1.7>. The sample mean is 38.567 and the sample standard deviation is 12.982.

Descriptive Statistics	Analysis Var (Age)
Observation	30
Missing Observations	0
Mean	38.000
Variance (n)	128.267
Variance (n-1)	132.690
Std Dev (n)	11.325
Std Dev (n-1)	11.519
Minimum	17.000
1st Quartile	29.250
Median	37.000
3rd Quartile	46.500
Maximum	67.000
Range	50.000
Interquartile Range	17.250
Coefficient of Variation (n)	29.80 %
Coefficient of Variation (n-1)	30.31 %

<Figure 11.1.7> Descriptive statistics of age

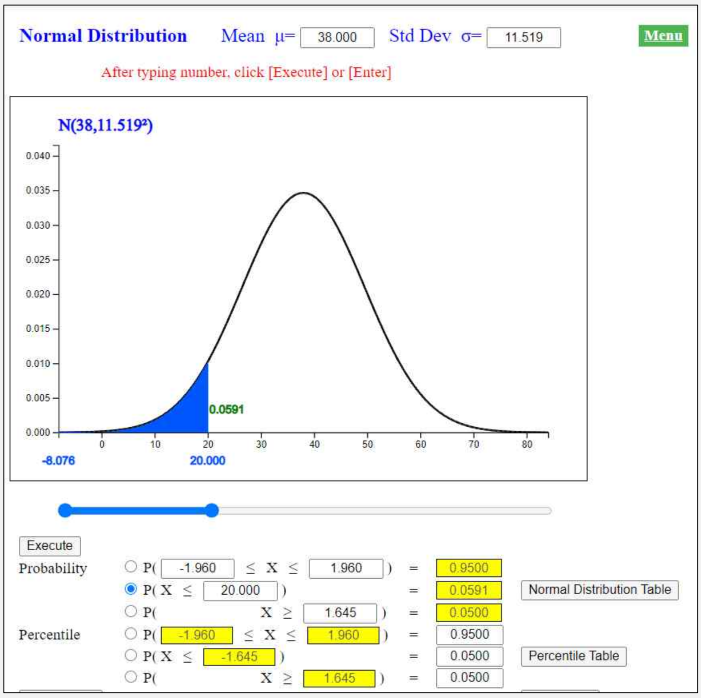
Example 11.1.2 Answer (continued)

Hence, the above hypothesis can be written in detail as follows:

H_0 : Sample data follow $N(38.000,\, 11.519^2)$.
H_1 : Sample data do not follow $N(38.000,\, 11.519^2)$.

- In order to find the expected frequency of each interval when H_0 is true, the expected probability of each interval is calculated first using the normal distribution $N(38.000,\, 11.519^2)$ as follows. The normal distribution module of『eStatU』makes it easy to calculate this probability of an interval. At the normal distribution module of『eStatU』, enter the mean of 38.000 and the standard deviation of 11.519. Click the second radio button of $P(X \le x)$ type and enter 20, then press the [Execute] button to calculate the probability as shown in <Figure 11.1.8>.

$$P(X < 20) = P(Z < \frac{20-38.000}{11.519}) = P(Z < -1.563) = 0.059$$



<Figure 11.1.8> Calculation of normal probability using『eStatU』

Similarly you can calculate the following probabilities.

$$P(20 \le X < 30) = P(\frac{20-38.000}{11.519} \le Z < \frac{30-38.000}{11.519}) = P(-1.563 \le Z < -0.695) = 0.185$$
$$P(30 \le X < 40) = P(\frac{30-38.000}{11.519} \le Z < \frac{40-38.000}{11.519}) = P(-0.695 \le Z < 0.174) = 0.325$$
$$P(40 \le X < 50) = P(\frac{40-38.000}{11.519} \le Z < \frac{50-38.000}{11.519}) = P(0.174 \le Z < 1.042) = 0.282$$
$$P(50 \le X < 60) = P(\frac{50-38.000}{11.519} \le Z < \frac{60-38.000}{11.519}) = P(1{,}042 \le Z < 1.910) = 0.121$$
$$P(X \ge 60) = P(Z \ge \frac{60-38.000}{11.519}) = P(Z \ge 1.910) = 0.028$$

- Expected frequency can be calculated by multiplying the sample size of 30 to the expected probability of each interval obtained above. The observed frequencies, expected probabilities, and expected frequencies for each interval can be summarized as the following table.

Example 11.1.2 Answer (continued)

Table 11.1.3 Observed and expected frequencies of each interval of $N(38.000,\ 11.519^2)$ distribution

Interval id	Interval	Observed frequency	Expected probability	Expected frequency
1	X < 20	2	0.059	1.77
2	20 $\leq$ X < 30	6	0.185	5.55
3	30 $\leq$ X < 40	8	0.325	9.75
4	40 $\leq$ X < 50	11	0.282	8.46
5	50 $\leq$ X < 60	2	0.121	3.63
6	60 $\leq$ X	1	0.028	0.84

- Since the expected frequencies of the 1st and 6th interval are less than 5, the intervals should be combined with adjacent intervals for testing the goodness of fit using the chi-square distribution as Table 11.1.4. The expected frequency of the last interval is still less than 5, but, if we combine this interval, there are only three intervals, we demonstrate the calculation as it is. Note that, due to computational error, the sum of the expected probabilities may not be exactly equal to 1 and the sum of the expected frequencies may not be exactly 30 in Table 11.1.4.

Table 11.1.4 Revised table after combining interval of small expected frequency

Interval id	Interval	Observed frequency	Expected probability	Expected frequency
1	X < 30	8	0.244	7.32
2	30 $\leq$ X < 40	8	0.325	9.75
3	40 $\leq$ X < 50	11	0.282	8.46
4	50 $\leq$ X	3	0.149	4.47
	Total	30	1.000	30.00

- The test statistic for the goodness of fit test is as follows:

$$\chi^2_{obs} = \frac{(8-7.32)^2}{7.32} + \frac{(8-9.75)^2}{9.75} + \frac{(11-8.46)^2}{8.46} + \frac{(3-4.47)^2}{4.47} = 1.623$$

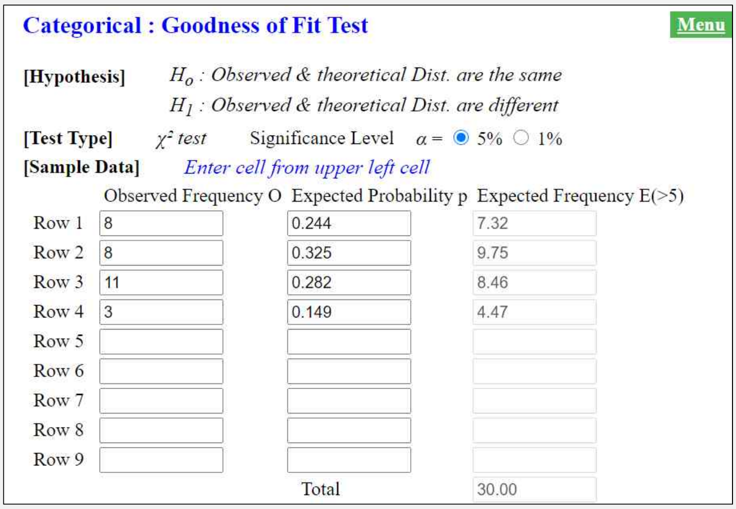
Since the number of intervals is 4, k becomes 4, and $m = 2$, because two population parameters μ and σ^2 are estimated from the sample data. Therefore, the critical value is as follows:

$$\chi^2_{k-m-1\,;\,\alpha} = \chi^2_{4-2-1\,;\,0.05} = \chi^2_{1\,;\,0.05} = 3.841$$

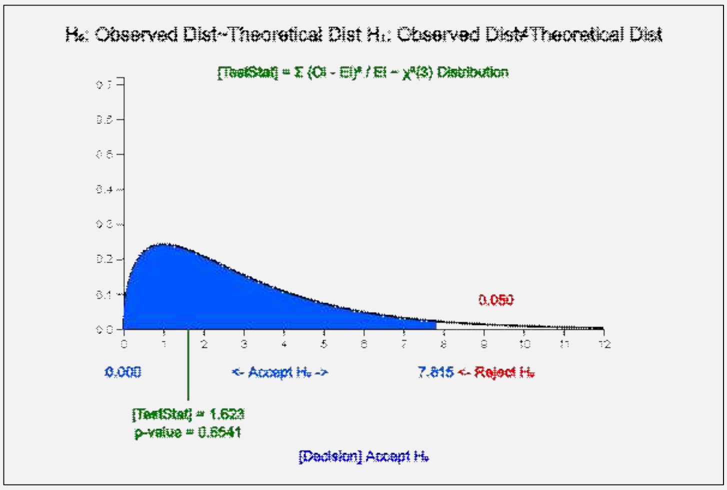
The observed test statistic is less than the critical value, we can not reject the null hypothesis that the sample data follows $N(38.000,\ 11.519^2)$.

- Test result can be verified using 'Goodness of Fit Test' in 『eStatU』. In the Input box that appears by selecting the 'Goodness of Fit Test' module, enter the data for 'observation frequency' and 'expected probability' in Table 11.1.4, as shown in <Figure 11.1.9>. After entering the data, select the significance level and press the [Execute] button to calculate the 'expected frequency' and produce a chi-square test result (<Figure 11.1.10>).

Example 11.1.2 Answer (continued)



<Figure 11.1.9> Data input for goodness of fit test in 『eStatU』



<Figure 11.1.10> Chi-square goodness of fit test using 『eStatU』

[Practice 11.1.2]

(Otter length)

Data of 30 otter lengths can be found at the following location of 『eStat』.

Ex ⇨ eBook ⇨ PR110102_OtterLength.csv.

Test the hypothesis that the population is normally distributed at the significance level of 5% using 『eStat』.

11.2 Testing Hypothesis for Contingency Table

- The contingency table or cross table discussed in Chapter 4 was a table that placed the possible values of two categorical variables in rows and columns, respectively, and examined frequencies of each cell in which the values of the two variables intersect. If this contingency table is for sample data taken from a population, it is possible to predict what would be the contingency table of the population. The test for the contingency table is usually an analysis of the relation between two categorical variables and it can be divided into the independence test and homogeneity test according to the sampling method for obtaining the data.

11.2.1 Independence Test

- The independence test of the contingency table is to investigate whether two categorical variables are independent when samples are extracted from one population. Consider the independence test with the following example.

Example 11.2.1

In order to investigate whether college students who are wearing glasses are independent by gender, a sample of 100 students was collected and its contingency table was prepared as follows:

Table 11.2.1 Wearing glasses by gender

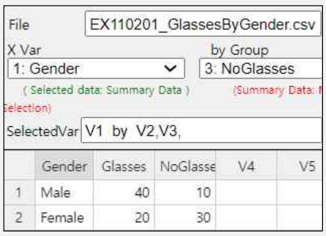
	Wear Glasses	No Glasses	Total
Men	40	10	50
Women	20	30	50
Total	60	40	100

Ex ⇨ eBook ⇨ EX110201_GlassesByGender.csv.

1) Using『eStat』, draw a line graph of the use of eyeglasses by men and women.
2) Test the hypothesis at 5% of the significance level to see if the gender variable and the wearing of glasses are independent or related to each other.
3) Check the result of the independence test using『eStatU』.

Answer

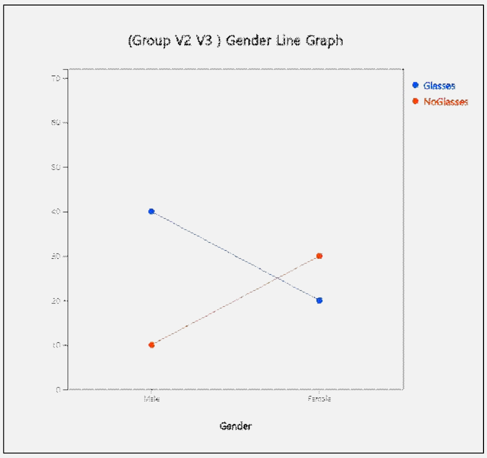
1) Enter data in 『eStat』 as shown in <Figure 11.2.1>.



<Figure 11.2.1> Data input

- Select 'Line Graph' icon from the main menu. If you click variables ‘Gender’, ‘Glasses’, ‘NoGlasses’ one by one, then a line graph as shown in <Figure 11.2.2> will appear in the Graph Area. If you look at the line graph, you can see that the ratio of wearing glasses for men and women are different. For men, there are many male students who wear glasses (80% of men) and many female students who do not wear glasses (60% of women). In such cases, the gender variable and the wearing of glasses are considered related. As such, when two variables are related, two lines of the line graph intersect to each other.

Example 11.2.1 Answer (continued)



<Figure 11.2.2> Line graph of wearing glasses by gender

2) If two variables are not related (i.e., if the two variables are independent of each other), the contingency table in Table 11.2.1 will show that the proportion of wearing glasses by men or women is equal to 60% which is the proportion of all students wearing glasses. In other words, if two variables are independent, the contingency table should be as follows:

Table 11.2.2 Contingency table when gender and wearing glasses are independent

	Wear Glasses	No Glasses	Total
Men	30	20	50
Women	30	20	50
Total	60	40	100

- If there is little difference between the observed contingency table and the contingency table in the case of independence, two categorical variables are said to be independent of each other. If the differences are very large, two categorical variables are related to each other. The independence test is a statistical method for determining that two categorical variables of the population are independent of each other by using the observed contingency table obtained from the sample. The independent test uses the chi-square distribution and the hypothesis is as follows:

  H_0 : Two variables of the contingency table are independent of each other.
  H_1 : Two variables of the contingency table are related.

- The test statistic for testing this hypothesis utilizes the difference between the observed frequency of the contingency table in the sample and the expected frequency of the contingency table when two variables are assumed to be independent which is similar to the goodness of fit test. The test statistic in this example is as follows:

$$\chi^2_{obs} = \frac{(40-30)^2}{30} + \frac{(10-20)^2}{20} + \frac{(20-30)^2}{30} + \frac{(30-20)^2}{20} \;=\; 16.67$$

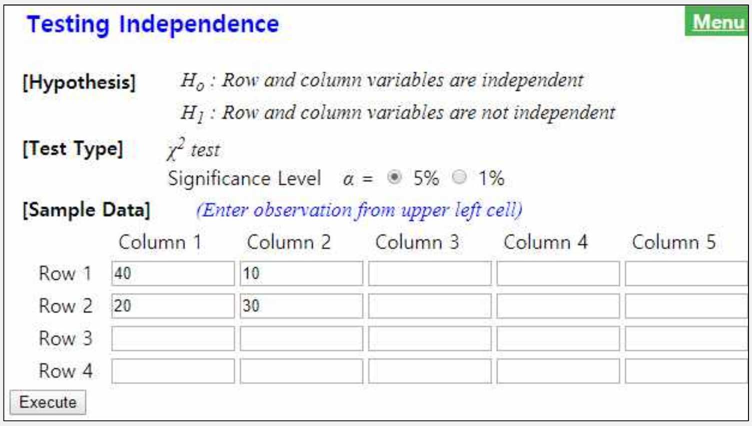
  This test statistic follows a chi-square distribution with $(r-1)(c-1)$ degrees of freedom where r is the number of rows (number of possible values of row variable) and c is the number of columns (number of possible values of column variable). Therefore, the decision rule to test the hypothesis is as follows:

Example 11.2.1 Answer (continued)

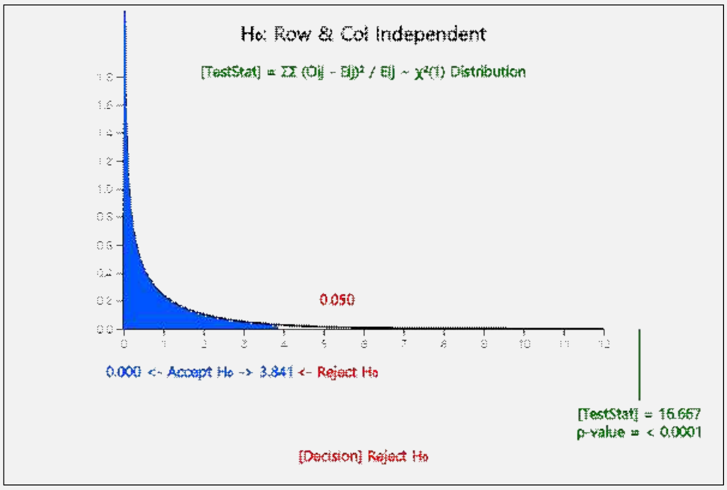
'If $\chi^2_{obs} > \chi^2_{(r-1)(c-1)\,;\,\alpha}$, then reject H_0.'

In this example, $\chi^2_{obs} = 16.67$ is greater than the critical value than $\chi^2_{(r-1)(c-1)\,;\,\alpha} = \chi^2_{(2-1)(2-1)\,;\,0.05} = \chi^2_{1\,;\,0.05} = 3.841$. Therefore, the null hypothesis that two variables are independent each other is rejected and we conclude that the gender and wearing glasses are related.

3) In the independence test of 『eStatU』, enter data as shown in <Figure 11.2.3> and press the [Execute] button to display the result of the chi-square test as shown in <Figure 11.2.4>.



<Figure 11.2.3> 『eStatU』 Test of Independence



<Figure 11.2.4> 『eStatU』 Chi-square test of independence

- Assume that there are r number of attributes of the variable A such as $A_1, A_2, \cdots, A_r$, and c number of attributes of the variable B such as $B_1, B_2, \cdots, B_c$. Let p_{ij} denote the probability of the cell of A_i and B_j attribute in the contingency table of A and B as Table 11.2.3. Here $p_{i.} = p_{i1} + p_{i2} + \cdots + p_{ic}$ denotes the probability of A_i and $p_{.j} = p_{1j} + p_{2j} + \cdots + p_{rj}$ denotes the probability of B_j.

Table 11.2.3 Notation of probabilities in $r \times c$ contingency table

		Variable B B_1 B_2 $\cdot\cdot\cdot\cdot$ B_c	Total
Vaiable A	A_1	p_{11} p_{12} $\cdot\cdot\cdot\cdot$ p_{1c}	$p_{1\cdot}$
	A_2	p_{21} p_{22} $\cdot\cdot\cdot\cdot$ p_{2c}	$p_{2\cdot}$
	$\cdot$	$\cdots$	$\cdot$
	$\cdot$	$\cdots$	$\cdot$
	$\cdot$	$\cdots$	$\cdot$
	A_r	p_{r1} p_{r2} $\cdot\cdot\cdot\cdot$ p_{rc}	$p_{r\cdot}$
Total		$p_{\cdot 1}$ $p_{\cdot 2}$ $\cdot\cdot\cdot\cdot$ $p_{\cdot c}$	1

- If two events A_i and B_j are independent, $P(A_i \cap B_j) = P(A_i) \cdot P(B_j)$ and hence, $p_{ij} = p_{i\cdot} \cdot p_{\cdot j}$. If two variables A and B are independent, all A_i and B_j should satisfy the above property which is called the independent test.

  H_0 : Variables A and B are independent.
  i.e. $p_{ij} = p_{i\cdot} \cdot p_{\cdot j}$ $i = 1, \cdots, r,$ $j = 1, \cdots, c$
  H_1 : Variables A and B are not independent.

- In order to test whether two variables of the population are independent, let us assume the observed frequencies, O_{ij}'s, of the contingency table from n samples are as follows:

Table 11.2.4 Observed frequency O_{ij} of $r \times c$ contingency table

		Variable B B_1 B_2 $\cdot\cdot\cdot\cdot$ B_c	Total
Variable A	A_1	O_{11} O_{12} $\cdot\cdot\cdot$ O_{1c}	$T_{1\cdot}$
	A_2	O_{21} O_{22} $\cdot\cdot\cdot$ O_{2c}	$T_{2\cdot}$
	$\cdot$	$\cdot$ $\cdot$ $\cdot\cdot\cdot$ $\cdot$	$\cdot$
	$\cdot$	$\cdot$ $\cdot$ $\cdot\cdot\cdot$ $\cdot$	$\cdot$
	$\cdot$	$\cdot$ $\cdot$ $\cdot\cdot\cdot$ $\cdot$	$\cdot$
	A_r	O_{r1} O_{r2} $\cdot\cdot\cdot$ O_{rc}	$T_{r\cdot}$
Total		$T_{\cdot 1}$ $T_{\cdot 2}$ $\cdot\cdot\cdot\cdot$ $T_{\cdot c}$	n

- If the null hypothesis H_0 is true, i.e., if two variables are independent of each other, the expected frequency of the sample data will be $n\ p_{i\cdot} p_{\cdot j}$. Since we do not know the population $p_{i\cdot}$ and $p_{\cdot j}$, if we use the estimates of $T_{i\cdot}/n$ and $T_{\cdot j}/n$, then the estimate of the expected frequency, E_{ij}, is as follows:

$$E_{ij} = n(\frac{T_{i\cdot}}{n})(\frac{T_{\cdot j}}{n}) = T_{i\cdot}(\frac{T_{\cdot j}}{n})$$

- The expected frequencies in case of independent can be explained that the proportions of each attribute of the B variable, $(T_{\cdot 1}/n, T_{\cdot 2}/n, \cdots, T_{\cdot r}/n)$, are maintained in each attribute of the A variable.

Table 11.2.5 Expected frequency E_{ij} of $r \times c$ contingency table

		Variable B			
		B_1	B_2	$\cdots$	B_c
Variable A	A_1	$E_{11} = T_{1.}\frac{T_{.1}}{n}$	$E_{12} = T_{1.}\frac{T_{.2}}{n}$	$\cdots$	$E_{1c} = T_{1.}\frac{T_{.c}}{n}$
	A_2	$E_{21} = T_{2.}\frac{T_{.1}}{n}$	$E_{22} = T_{2.}\frac{T_{.2}}{n}$	$\cdots$	$E_{2c} = T_{2.}\frac{T_{.c}}{n}$
	$\vdots$	$\vdots$			
	A_r	$E_{r1} = T_{r.}\frac{T_{.1}}{n}$	$E_{r2} = T_{r.}\frac{T_{.2}}{n}$	$\cdots$	$E_{rc} = T_{r.}\frac{T_{.c}}{n}$

- The test statistic utilizes the difference between O_{ij} and E_{ij} as follows:

$$\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}}$$

This test statistic follows approximately a chi-square distribution with $(r-1)(c-1)$ degrees of freedom. Therefore, the decision rule to test the hypothesis with significance level of α is as follows:

$$\text{‘If } \chi^2_{obs} = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} > \chi^2_{(r-1)(c-1);\alpha}, \text{ then reject } H_0\text{’}$$

Independence Test

Hypothesis:

H_0 : Variables A and B are independent.

i.e., $p_{ij} = p_{i.} \cdot p_{.j} \quad i = 1, \cdots, r, \quad j = 1, \cdots, c$

H_1 : Variables A and B are not independent.

Decision Rule:

$$\text{‘If } \chi^2_{obs} = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} > \chi^2_{(r-1)(c-1);\alpha}, \text{ then reject } H_0\text{’}$$

where r is the number of attributes of row variable and c is the number of attributes of column variable.

☞

In order to use the chi-square distribution for the independence test, all expected frequencies are at least 5 or more.

If an expected frequency of a cell is smaller than 5, the cell is combined with adjacent cell for analysis.

- Consider an example of the independent test with many rows and columns.

Example 11.2.2

A market research institute surveyed 500 people on how three beverage products (A, B and C) are preferred by region and obtained the following contingency table.

Table 11.2.6 Survey for preference of beverage by region

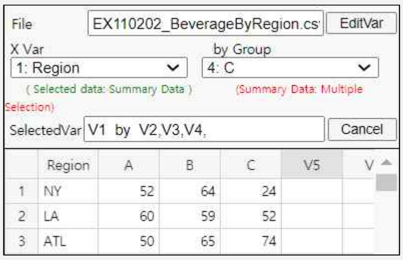
		Beverage A	B	C	Total
Region	New York	52	64	24	140
	Los Angels	60	59	52	171
	Atlanta	50	65	74	189
Total		162	188	150	500

Ex ⇨ eBook ⇨ EX110202_BeverageByRegion.csv.

1) Draw a line graph of beverage preference by region using 『eStat』 and analyze the graph.
2) Test whether the beverage preference by the region is independent of each other at the significance level of 5%.
3) Check the result of the independence test using 『eStatU』 .

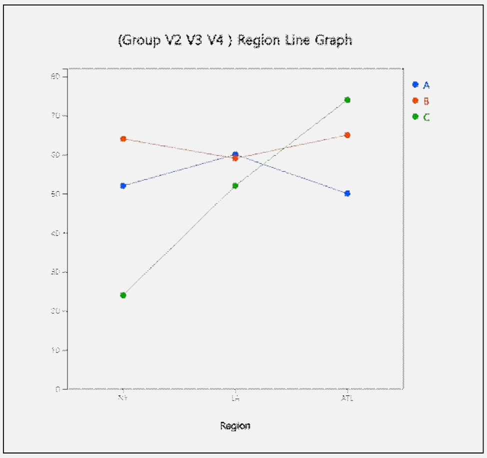
Answer

1) Enter the data in 『eStat』 as shown in <Figure 11.2.5>.



<Figure 11.2.5> Data input

- Select 'Line Graph' and click variables ‘Region’, ‘A’, ‘B’, and ‘C’ in order, then the line graph shown in <Figure 11.2.6> will appear. If you look at the line graph, you can see the cross-section of the lines from region to region, and the regional preference is different. Can you statistically conclude that the region and beverage preference are related?



<Figure 11.2.6> Line graph by region and beverage

2) The hypothesis for the independence test is as follows:

H_0 : Region and beverage preference are independent.
H_1 : Region and beverage preference are not independent.

Example 11.2.2 Answer (continued)

- In order to calculate the expected frequencies, we first calculate the proportions of each beverage preference without considering the region as follows:

$$\left(\frac{162}{500},\ \frac{88}{500},\ \frac{50}{500}\right)$$

- If two variables are independent, these proportions should be kept in each region. Hence, the expected frequencies in each region can be calculated as follows:

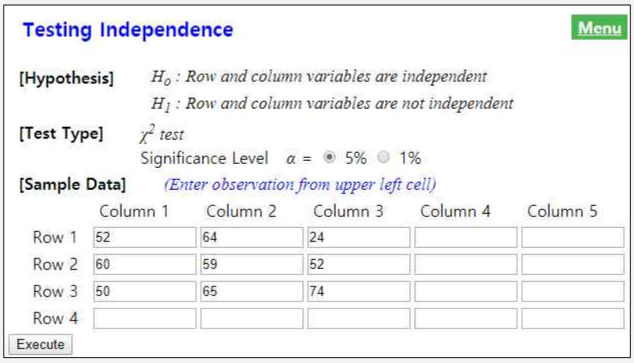
$$E_{11} = 140 \times \frac{162}{500} = 45.36 \quad E_{12} = 140 \times \frac{188}{500} = 52.64 \quad E_{13} = 140 \times \frac{150}{500} = 42.00$$
$$E_{21} = 171 \times \frac{162}{500} = 55.40 \quad E_{22} = 171 \times \frac{188}{500} = 64.30 \quad E_{23} = 171 \times \frac{150}{500} = 51.30$$
$$E_{31} = 189 \times \frac{162}{500} = 61.24 \quad E_{32} = 189 \times \frac{188}{500} = 71.06 \quad E_{33} = 189 \times \frac{150}{500} = 56.70$$

- The chi-square test statistic and critical value are as follows:

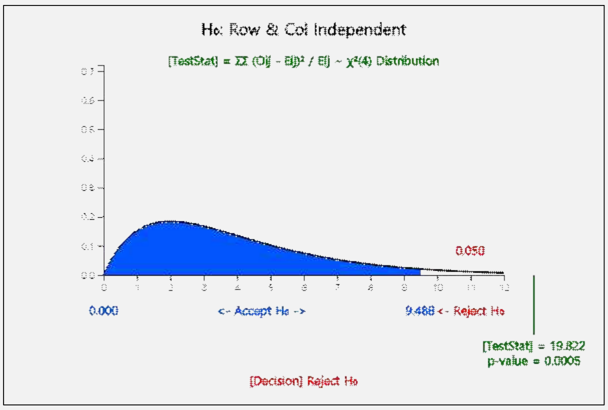
$$\chi^2_{obs} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{(O_{ij}-E_{ij})^2}{E_{ij}} = \frac{(52-45.36)^2}{45.36} + \frac{(60-55.40)^2}{55.40} + \cdots + \frac{(74-56.70)^2}{56.70} = 18.825$$
$$\chi^2_{(r-1)(c-1)\,;\,\alpha} = \chi^2_{(3-1)(3-1)\,;\,0.05} = \chi^2_{4\,;\,0.05} = 9.488$$

Therefore, the null hypothesis H_0 is rejected at the significance level of 5% and conclude that the region and beverage are related.

3) In the independence test of『eStatU』, enter data as shown in <Figure 11.2.7> and click the [Execute] button to display the result of the chi-square test as shown in <Figure 11.2.8>.

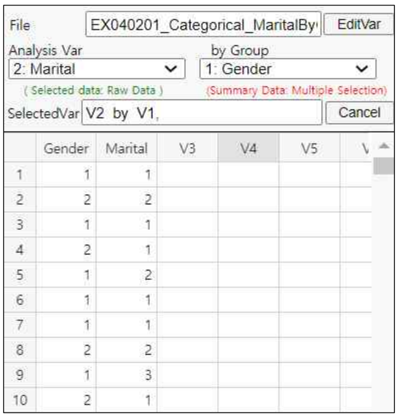


<Figure 11.2.7> Data input for Independence Test at『eStatU』



<Figure 11.2.8> Chi-square Independence Test at『eStatU』

- As described in Chapter 4, if a contingency table is made using raw data (<Figure 11.2.9>), 『eStat』 provides the result of the independence test as shown in <Figure 11.2.10>. In this case, if a cell of the contingency table has a small expected number, the test result should be interpreted carefully.



<Figure 11.2.9> Raw data input for independence test

Cross Table	Col Variable	(Gender)			
Row Variable (Marital)	1	2	Total		
Group 1	4 66.7%	2 33.3%	6 100%		
Group 2	1 33.3%	2 66.7%	3 100%		
Group 3	1 100.0%	0 0.0%	1 100%		
Total	6 60.0%	4 40.0%	10 100%		
	Missing Observations	0			
Independence Test					
Sum of χ^2 value	1.667	deg of freedom	2	p-value	0.4346

<Figure 11.2.10> 『eStat』 contingency table and independence test

[Practice 11.2.1]

A guidance counselor surveyed 100 high school students for reading and watching TV. The following table was obtained by classifying each item as high and low. Using the significance level of 0.05, are these data sufficient to claim that the reading and TV viewing are related? Check the test result using 『eStatU』 .

	Reading		Total
	High	Low	
TV viewing High	40	18	58
TV viewing Low	31	11	42
Total	71	29	100

Ex ⇨ eBook ⇨ EX110201_TV_Reading.csv.

11.2.2 Homogeneity Test

- The independence test described in the previous section were for the contingency table of two categorical variables based on sample data from one population. However, similar contingency table may be taken from several populations, where each sample is drawn from such a different population. It can often be seen when the research is more efficiently to be done or when time and space constraints are imposed. For example, if you want to compare the English scores of freshman, sophomore, junior and senior students in a university, it is reasonable to take samples from each grade and analyze them. In this case, the contingency table is as follows:

Table 11.2.7 A contingency table of English score by grade level

		Freshman	Sophomore	Junior	Senior
English score	A	-	-	-	-
	B	-	-	-	-
	C	-	-	-	-
	D	-	-	-	-

- If this contingency table is derived from each grade population, the question we are curious is not an independence of the English score and grade level, but four distributions of English scores are equal. The hypothesis for a contingency table of samples drawn from multiple populations is as follows. It is called the **homogeneity test**.

H_0 : Distributions of several populations for a categorical variable are homogeneous.
H_1 : Distributions of several populations for a categorical variable are not homogeneous.

- The test statistic for the homogeneity test is the same as the independence test as follows:

$$\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}}$$

Here r is the number of attributes of the categorical variable and c is the number of populations.

Homogeneity Test

Hypothesis:

H_0 : Several population distributions for a categorical variable are homogeneous.
H_1 : Several population distributions for a categorical variable are not homogeneous.

Decision Rule:

If $\chi^2_{obs} = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} > \chi^2_{(r-1)(c-1);\alpha}$, then reject H_0

Here r is the number of attributes of the categorical variable and c is the number of populations.

☞ **In order to use the chi-square distribution for the homogeneity test, all expected frequencies are at least 5 or more.**

If an expected frequency of a cell is smaller than 5, the cell is combined with adjacent cell for analysis.

Example 11.2.3

In order to investigate whether viewers of TV programs are different by age for three programs (A, B and C), 200, 100 and 100 samples were taken separately from the population of young people(20s), middle-aged people (30s and 40s), and older people (50s and over) respectively. Their preference of the program were summarized as follows. Test whether TV program preferences vary by age group at the significance level of 5%.

Table 11.2.8 Preference of TV program by age group

		Young	Middle Aged	Older	Total
TV Program	A	120	10	10	140
	B	30	75	30	135
	C	50	15	60	125
Total		200	100	100	400

Answer

- The hypothesis of this problem is as follows:

H_0 : TV program preferences for different age groups are homogeneous.
H_1 : TV program preferences for different age groups are not homogeneous.

- Proportions of the number of samples for each age group are as follows:

$$\left(\frac{200}{400}, \frac{100}{400}, \frac{100}{400}\right)$$

Therefore, the expected frequencies of each program when H_0 is true are as follows:

$$E_{11} = 140 \times \frac{200}{400} = 70 \quad E_{12} = 140 \times \frac{100}{400} = 35 \quad E_{13} = 140 \times \frac{100}{400} = 35$$

$$E_{21} = 135 \times \frac{200}{400} = 67.5 \quad E_{22} = 135 \times \frac{100}{400} = 33.75 \quad E_{23} = 135 \times \frac{100}{400} = 33.75$$

$$E_{31} = 125 \times \frac{200}{400} = 62.5 \quad E_{32} = 125 \times \frac{100}{400} = 31.25 \quad E_{33} = 125 \times \frac{100}{400} = 31.25$$

- Test statistic and critical value are as follows:

$$\chi^2_{obs} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} = \frac{(120-70)^2}{70} + \frac{(10-35)^2}{35} + \cdots + \frac{(60-31.25)^2}{31.25} = 180.495$$

$$\chi^2_{(r-1)(c-1)\,;\,\alpha} = \chi^2_{(3-1)(3-1)\,;\,0.05} = \chi^2_{4\,;\,0.05} = 9.488$$

Since χ^2_{obs} is greater than the critical value, H_0 is rejected. TV programs have different preferences for different age groups.

[Practice 11.2.2]

To evaluate the effectiveness of typing training, 100 documents by company employees who received type training and 100 documents by employees who did not receive typing training were evaluated. Evaluated documents are classified as good, normal, and low. The following table shows a classification of the evaluation for total 200 documents according to whether or not they received training. Test the null hypothesis that distributions of the document evaluation are the same in both populations. Use $\alpha = 0.05$ and check your test result using 『eStatU』.

Document Evaluation	Training		Total
	Typing training	No typing training	
Good	48	12	60
Normal	39	26	65
Low	13	62	75
Total	100	100	200

Exercise

11.1 300 customers selected randomly are asked on which day of the week they usually went to the grocery store and received the following votes. Can you conclude that the percentage of days customers prefer is different? Use the 5% significance level. Check the test result using 『eStatU』.

Day	Mon	Tue	Wed	Thr	Fri	Sat	Sun	Total
Number of Customers	10	20	40	40	80	60	50	300

11.2 The market shares of toothpaste brands A, B, C and D are known to be 0.3, 0.6, 0.08, and 0.02 respectively. The result of a survey of 600 people for the toothpaste brands are as follows. Can you conclude from these data that the existing market share is incorrect? Use $\alpha = 0.05$ and check your test result using 『eStatU』.

Brand	A	B	C	D	Total
Number of Customers	192	342	44	22	600

11.3 The following table shows the distribution by score by conducting an aptitude test on 223 workers at a plant. The mean and variance from the sample data are 75 and 386 respectively. Test whether the scores of the aptitude test follow a normal distribution. Use $\alpha = 0.05$ and check your test result using 『eStatU』.

Score interval	Number of Workers
$X < 40$	10
$40 \le X < 50$	12
$50 \le X < 60$	17
$60 \le X < 70$	37
$70 \le X < 80$	55
$80 \le X < 90$	51
$90 \le X < 100$	34
$X \ge 100$	7
Total	223

11.4 The following data shows the highest temperature of a city during the month of August. Test whether the temperature data follow a normal distribution with the 5% significance level. (Unit: °C)

29, 29, 34, 35, 35, 31, 32, 34, 38, 34, 33, 31, 31, 30, 34, 35,
34, 32, 32, 29, 28, 30, 29, 31, 29, 28, 30, 29, 29, 27, 28.

11.5 For market research, a company obtained data on the educational level and socio-economic status of 375 housewives and summarized a contingency table as follows. Test the null hypothesis that social and economic status and educational level are independent at the significance level of 0.05. Check the test result using 『eStatU』.

Socio-economic status	Education Level					Total
	Elementary	Middle	High	College	Above	
1	10	7	3	4	1	25
2	14	10	7	4	2	37
3	9	25	13	18	3	68
4	7	9	38	44	6	104
5	3	8	14	18	62	105
6	2	3	8	10	13	36
Total	45	62	83	98	87	375

11.6 Government agencies surveyed workers who wanted to get a job and classified 532 respondents according to the gender and technical level as follows. Does these data provide sufficient evidence that the technical level and gender are related? Use $\alpha = 0.05$ and check your test result using 『eStatU』.

Technical Level	Gender		Total
	Male	Female	
Skilled worker	106	6	112
Semi-skilled worker	93	39	132
Unskilled worker	215	73	288
Total	414	118	532

11.7 A guidance counselor surveyed 110 high school students for reading and watching TV. The following table was obtained by classifying each item as high and low. At the significance level of 0.05, are these data sufficient to claim that the reading and TV viewing are related? Check the test result using 『eStatU』.

	Reading		Total
	High	Low	
TV viewing High	40	18	58
TV viewing Low	41	11	52
Total	81	29	110

11.8 165 defective products produced in two plants operated by the same company were classified depending on whether they were due to low occupational awareness or low quality raw materials by each plant. Test the null hypothesis that the cause of the defect and production plant are independent with the significance level of 0.05. Check the test result using 『eStatU』.

Cause of defect	Plant		Total
	A	B	
low occupational awareness	21	72	93
low quality raw materials	46	26	72
Total	67	98	165

11.9 To evaluate the effectiveness of typing training, 110 documents by company employees who received type training and 120 documents by employees who did not receive typing training were evaluated. Evaluated documents are classified as good, normal, and low. The following table shows a classification of the evaluation for total 230 documents according to whether or not they received training. Test the null hypothesis that typing training and document evaluation are independent. Use $\alpha = 0.05$ and check your test result using 『eStatU』.

	Evaluation			Total
	Good	Normal	Low	
Typing training	48	39	23	110
No typing training	12	36	72	120
Total	60	75	95	230

11.10 A company with three large plants applied different working conditions and wage systems to three plants to ask them for satisfaction with the new system six months later. 250 workers from each of three plants were randomly selected and the survey results were as follows. Is there sufficient evidence that workers at each plant have different satisfaction levels? Test with the significance level of 0.05. Check the test result using 『eStatU』.

Plant	Job Satisfaction				Total
	Very satisfied	Satisfied	Average	Not satisfied	
Plant 1	135	70	25	20	250
Plant 2	145	80	15	10	250
Plant 3	140	75	20	15	250
Total	420	225	60	45	750

Multiple Choice Exercise

11.1 What tests do you need to investigate whether the sample data follow a theoretical distribution?

① Goodness of fit test ② Independence test
③ Test for population proportion ④ Test for two population means

11.2 In order to test whether sample data of a continuous variable follow a distribution, what is the first necessary work for the goodness of fit test?

① log transformation ② frequency distribution of interval
③ [0,1] transformation ④ frequency distribution

11.3 How do you test the hypothesis that the two categorical variables of a sample from a population have no relation?

① Goodness of fit test ② Independence test
③ Test for population proportion ④ Test for homogeneity

11.4 How do you test the hypothesis that the samples from two categorical populations have the same distribution?

① Goodness of fit test ② Independence test
③ Test for population proportion ④ Test for homogeneity

11.5 Which of the following statistical distributions is used to test for a contingency table?

① t distribution ② χ^2 distribution
③ binomial distribution ④ Normal distribution

(Answers)
11.1 ①, 11.2 ②, 11.3 ②, 11.4 ④, 11.5 ②