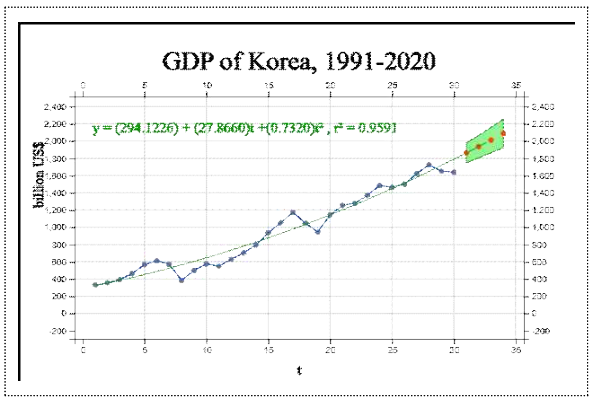
13

Time Series Analysis



SECTIONS

13.1 What is Time Series Analysis?
13.2 Smoothing of Time Series
13.3 Transformation of Time Series
13.4 Regression Model and Forecasting
13.5 Exponential Smoothing Model and Forecasting
13.6 Seasonal Model and Forecasting

CHAPTER OBJECTIVES

In this chapter, we study data observed over time, time series, and find out about:

- What is time series analysis and what are the types of time series models?
- How to smooth a time series.
- How to transform a time series.
- Prediction method using regression model.
- Prediction method using exponential smoothing model.
- How to predict future values with models for seasonal time series..

We will be mainly focused on descriptive methods and simple models, and discussion of the Box-Jenkins model and other theoretical models will not be discussed.

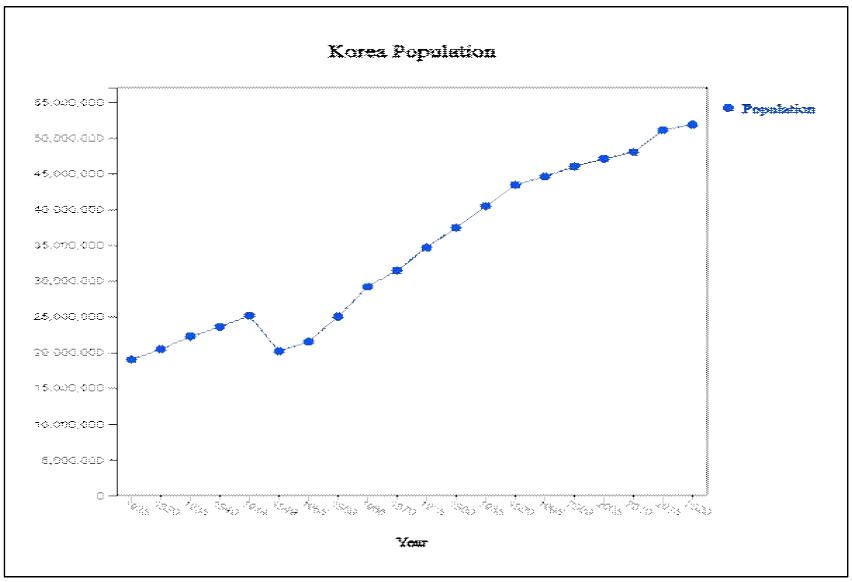
13.1 What is Time Series Analysis?

- **Time series** data refers to data recorded according to changes in time. In general, observations are made at regular time intervals such as year, season, month, or day, and this is called a **discrete time series**. There may be time series that are continuously observed, but this book will only deal with the analysis of discrete time series.
- An example of a discrete time series is the population of Korea as shown in [Table 13.1.1]. This data is from the census conducted every five years in Korea from 1925 to 2020 (except for 1944 and 1949). .

[Table 13.1.1] Population of Korea
(Source: Korea National Statistical Office, Census till 2010, Registered Census 2015, 2020)

Year	Population	Year	Population	Year	Population
1925	19,020,030	1960	24,989,241	1995	44,553,710
1930	20,438,108	1966	29,159,640	2000	45,985,289
1935	22,208,102	1970	31,435,252	2005	47,041,434
1940	23,547,465	1975	34,678,972	2010	47,990,761
1944	25,120,174	1980	37,406,815	2015	51,069,375
1949	20,166,756	1985	40,419,652	2020	51,829,136
1955	21,502,386	1990	43,390,374		

- As shown in the table above, it is not easy to understand the overall shape of the time series displayed in numbers. The first step in time series analysis is to observe the time series by drawing a time series plot with the X axis as time and the Y axis as time series values. For example, the time series plot of the total population in Korea is shown in <Figure 13.1.1>.



<Figure 13.1.1> Time Series of Korea Population

- Observing this figure, Korea's population has an overall increasing trend, but the population decreased sharply in 1944-1949 due to World War II. It can be seen that the population expanded rapidly after the Korean war in 1953 and slowed since 1990. It can also be seen that the growth has slowed further in the last 10 years. By observing the time series in this way, trends, change points, and outliers

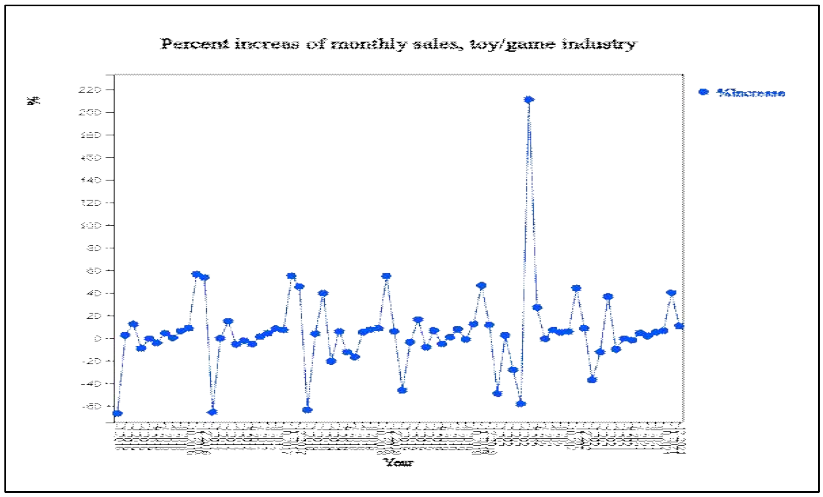
can be observed, which is helpful in selecting an analysis model or method suitable for the data.

- Time series that we frequently encounter include monthly sales of department stores and companies, daily composite stock index, annual crop production, yearly export and import time series, and yearly national income and economic growth rate, and so on.

- [Table 13.1.2] shows the percent increase in monthly sales of the US toy/game industry for the past 6 years, and <Figure 13.1.2> is a plot of this time series. As it is the rate of change from the previous month, it can be observed that it is seasonal data showing a large increase in November and December every year, moving up and down based on 0. However, May 2020 is an extreme with an increase rate of 211% unlike other years. For time series, you can better examine the characteristics of the data by converting the raw time series into the rate of change.

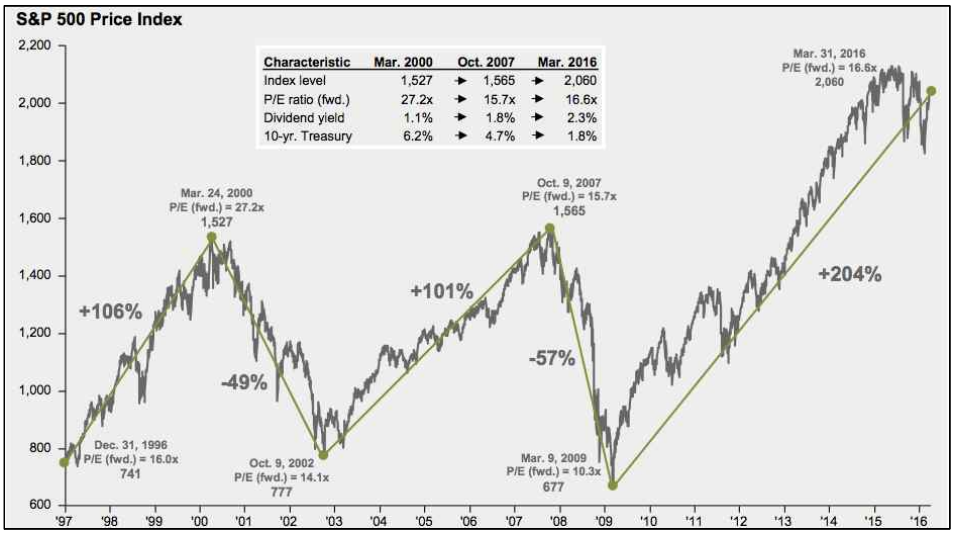
[Table 13.1.2] Percent Increase, Monthly Sales of Toy/Game in US(%)
(Source: Bureau of Census, US)

Year.Month	Percent Increase	Year.Month	Percent Increase	Year.Month	Percent Increase
2016.01	-66.7	2018.01	-63.6	2020.01	-49.1
2016.02	2.5	2018.02	3.6	2020.02	2.2
2016.03	12.5	2018.03	39.8	2020.03	-28.2
2016.04	-9.0	2018.04	-21.0	2020.04	-58.2
2016.05	-0.6	2018.05	5.9	2020.05	211.1
2016.06	-4.4	2018.06	-12.4	2020.06	26.8
2016.07	4.3	2018.07	-16.9	2020.07	-0.8
2016.08	0.0	2018.08	5.2	2020.08	7.0
2016.09	6.1	2018.09	7.5	2020.09	4.9
2016.10	8.6	2018.10	8.5	2020.10	5.8
2016.11	56.4	2018.11	54.9	2020.11	44.1
2016.12	53.6	2018.12	5.8	2020.12	8.5
2017.01	-65.6	2019.01	-46.2	2021.01	-37.1
2017.02	-0.1	2019.02	-3.8	2021.02	-12.2
2017.03	14.7	2019.03	16.3	2021.03	37.0
2017.04	-5.7	2019.04	-8.4	2021.04	-10.3
2017.05	-2.4	2019.05	6.6	2021.05	-0.5
2017.06	-5.5	2019.06	-5.3	2021.06	-2.0
2017.07	1.3	2019.07	0.8	2021.07	4.6
2017.08	4.2	2019.08	7.7	2021.08	1.8
2017.09	8.4	2019.09	-1.2	2021.09	5.2
2017.10	7.2	2019.10	12.2	2021.10	6.4
2017.11	54.9	2019.11	46.7	2021.11	40.0
2017.12	45.5	2019.12	11.7	2021.12	10.6



<Figure 13.1.2> Percent Increase, Monthly Sales of Toy/Game in US(%)

- Most time series have four components: trend, season, cycle, and other irregular factors. A **trend** is a case in which a time series has a certain trend, such as a line or a curved shape as time elapses, and there are various types of trends. Trends can be understood as a consumption behavior, population variations, and inflation that appear in time series over a long period of time. **Seasonal** factors are short-term and regular fluctuation factors that exist quarterly, monthly, or by day of the week. Time series such as monthly rainfall, average temperature, and ice cream sales have seasonal factors. Seasonal factors generally have a short cycle, but fluctuations when the cycle occurs over a long period of time rather than due to the season is called a **cycle** factor. By observing these cyclical factors, it is possible to predict the boom or recession of a periodic economy. {Figure 13.1.3} shows the US S&P 500 Index from 1997 to 2016, and a six-year cycle can be observed.



<Figure 13.1.3>] US S&P500 Index (1997- 2016)

- Other factors that cannot be explained by trend, season, or cyclical factors are called **irregular** or **random** factors, which refer to variable factors that appear due to random causes regardless of regular movement over time.

13.1.1 Time Series Model

- By observing the time series, you can predict how this time series will change in the future by building a time series model that fits the probabilistic characteristics of this data. Because the time series observed in reality has a very diverse form, the time series model is also very diverse, from simple to very complex. In general, time series models for a single variable can be divided into the following four categories.

A. Regression Model

A model that explains data or predicts the future by expressing a time series in the form of a function related to time is the most intuitive and easy to understand model. That is, when a time series is an observation of a random variable, $Y_1, Y_2, \cdots, Y_n$, it is expressed as the following model:

$$Y_t = f(t) + \epsilon_t, \qquad t = 1, 2, \cdots, n$$

Here ϵ_t is the error of the time series that cannot be explained by a function $f(t)$. In general ϵ_t is assumed independent, $E(\epsilon_t) = 0$, and $Var(\epsilon_t)$ = σ^2 which is called a white noise. For example, the following model can be applied to a time series in which the data is horizontal or has a linear trend.

Horizontal: $Y_t = \mu + \epsilon_t$
Linear Trend: $Y_t = a + bt + \epsilon_t$

B. Decomposition Model

The model that decomposes the time series into four factors, i.e., trend(T_t), cycle(C_t), seasonal(S_t), and irregular(I_t), is an analysis method that has been used for a long time based on empirical facts. It can be divided into additive model and multiplicative model.

Additive Model: $Y_t = T_t + C_t + S_t + I_t$
Multiplicative Model: $Y_t = T_t \cdot C_t \cdot S_t \cdot I_t$

Here T_t, C_t, S_t are deterministic function, I_t is a random variable. If we take the logarithm of a multiplicative model, it becomes an additive model. If the number of data is not enough the cycle factor can be omitted in the model.

C. Exponential Smoothing Model

Time series data are often more related to recent data than to past data. The above two types of models are models that do not take into account the relationship between the past time series data and the recent time series data. Models using moving averages and exponential smoothing are often used to explain and predict data using the fact that time series forecasting is more related to recent data.

D. Box-Jenkins ARIMA Model

The above models are not methods that can be applied to all types of time series, and the analyst selects and applies them according to the type of data. Box and Jenkins presented the following general ARIMA model that can be

applied to all time series of stationary or nonstationary type as follows:

$$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots$$

The ARIMA model considers the observed time series as a sample extracted from a population time series, studies the probabilistic properties of each model, and establishes an appropriate time series model through parameter estimation and testing. For the ARIMA model, autocorrelation coefficients between time lags are used to identify a model. The ARIMA model is beyond the scope of this book, so interested readers are encouraged to consult the bibliography.

- In the above time series model, the regression model and ARIMA model are systematic models based on statistical theory, and the decomposition model and exponential smoothing model are methods based on experience and intuition. In general, regression models using mathematical functions and models using decomposition are known to be suitable for predicting slow-changing time series, whereas exponential smoothing and ARIMA models are known to be effective in predicting very rapidly changing time series.
- For all time series models, it is impossible to predict due to sudden changes. And because time series has so many different forms, it cannot be said that one time series model is always superior to another. Therefore, rather than applying only one model to a time series, it is necessary to establish and compare several models, combine different models, or make an effort to determine the final model by combining opinions of experts familiar with the time series.

13.1.2 Evaluation of Time Series Model

- Let the time series be the observed values of the random variables $Y_1, Y_2, \cdots, Y_n$ and $\hat{Y}_1, \hat{Y}_2, \cdots, \hat{Y}_n$ be the values predicted by the model. If the model agrees exactly, the observed and predicted values are the same, and the model error ϵ_t is zero. In general, it is assumed that the error ϵ_t's of the time series model are independent random variables which follow the same normal distribution with a mean of 0 and a variance of σ^2. The accuracy of a time series model can be evaluated using residual, $Y_t - \hat{Y}_t$, which is a measure by subtracting the predicted value from the observed value. In general, the following mean squared error (MSE) is commonly used for the accuracy of a model and the smaller the MSE value, the more appropriate the predicted model is judged.

$$\mathrm{MSE} = \frac{\sum_{t=1}^{n} \left(\mathrm{Y}_t - \hat{\mathrm{Y}}_t\right)^2}{\mathrm{n}}$$

The mean square error is used as an estimator for the variance σ^2 of the error. Since MSE can have a large value, the root mean squared error (RMSE) is often used.

$$RMSE = \sqrt{MSE}$$

13.2 Smoothing of Time Series

- Original time series data can be used to make a time series model by observing trends, but in many cases, time series can be observed by **smoothing** to understand better**.** In a time series such as stock price, it is often difficult to find

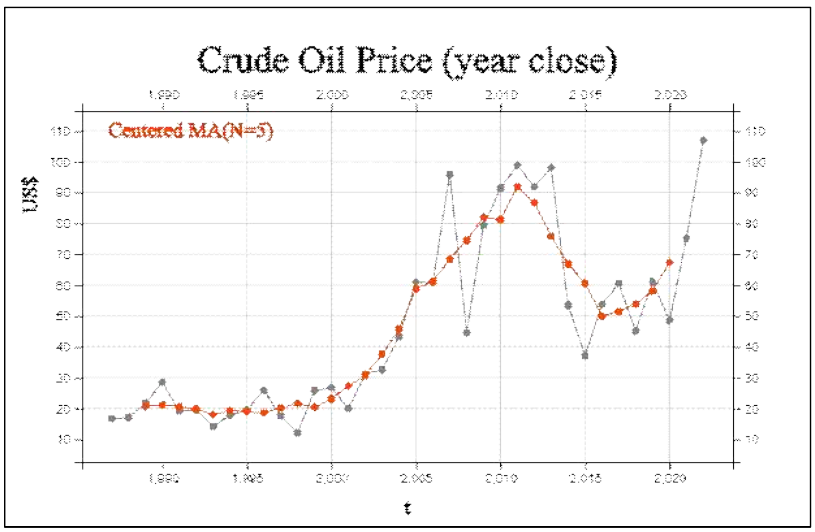
a trend because of temporary or short-term fluctuations due to accidental coincidences or cyclical factors. In this case, smoothing techniques are used as a method to effectively grasp the overall long-term trend by removing temporary or short-term fluctuations. The centered moving average method and the exponential smoothing method are widely used.

13.2.1 Centered Moving Average

- The time series in [Table 13.2.1] is the world crude oil price based on the closing price every year from 1987 to 2022. Looking at <Figure 13.2.1>, it can be seen that the short-term fluctuations in the time series are large. However, causes such as oil shocks are short-term and not continuous, so if we are interested in the long-term trend of gasoline consumption, it would be more effective to look at the fluctuations caused by short-term causes.

[Table 13.2.1] Price of Crude Oil (End of Year Price, US$) and 5-point Moving Average

Year	Price of Oil	5-point Moving Average	Year	Price of Oil	5-point Moving Average
1987	16.74		2005	61.06	58.746
1988	17.12		2006	60.85	61.164
1989	21.84	20.666	2007	95.95	68.370
1990	28.48	21.216	2008	44.60	74.434
1991	19.15	20.630	2009	79.39	82.030
1992	19.49	19.816	2010	91.38	81.206
1993	14.19	18.028	2011	98.83	91.920
1994	17.77	19.378	2012	91.83	86.732
1995	19.54	19.010	2013	98.17	75.882
1996	25.90	18.600	2014	53.45	66.866
1997	17.65	20.198	2015	37.13	60.592
1998	12.14	21.634	2016	53.75	49.988
1999	25.76	20.446	2017	60.46	51.526
2000	26.72	23.158	2018	45.15	53.804
2001	19.96	27.232	2019	61.14	58.096
2002	31.21	30.752	2020	48.52	67.394
2003	32.51	37.620	2021	75.21	
2004	43.36	45.798	2022	106.95	



<Figure 13.2.1> Price of Crude Oil and 5-point Moving Average

- The **N-point centered moving average** of a time series refers to the average of N data from a single point in time. For example, in crude oil price data, the value of the five-point moving average for a specific year is the average of the data for two years before the specific year, that year, and the data for the next two years. Expressed as an expression, if M_t is a moving average in time t, the 5-point centered moving average is as follows:

$$M_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{5}$$

For example, the 5-point centered moving average for 1989 is as follows.

$$\begin{aligned} M_{1989} &= \frac{Y_{1987} + Y_{1988} + Y_{1989} + Y_{1990} + Y_{1991}}{5} \\ &= \frac{16.74 + 17.12 + 21.84 + 28.48 + 19.15}{5} \\ &= 20.6660 \end{aligned}$$

[Table 13.2.2] shows the values of all 5-points centered moving averages obtained in this way and <Figure 13.2.1> is the graph of 5-points moving average. Note that the moving averages for the first two years and the last two years cannot be obtained here. It can be seen that the graph of the moving average is better for grasping the long-term trend than the graph of the original data because short-term fluctuations are removed.

- The choice of a value N for the N-points moving average is important. A large value of N will provide a smoother moving average, but it has the disadvantage of losing more points at both ends and insensitive to detecting important trend changes. On the other hand, if you choose small N, you will lose less data at both ends, but you may not be able to get the smoothing effect because you will not sufficiently eliminate short-term fluctuations. In general, try a few values N to reflect important changes that should not be missed, while achieving a smoothing effect and balancing the points not to lose too much at both ends.
- If the value of N is an even number, there is a difficulty in obtaining a central moving average with the same number of data on both sides of the base year. For example, the center of the four-point moving average from 1987 to 1990 is between 1988 and 1989. If you denote this as $M_{1988.5}$, it can be calculated as

follows:

$$\begin{aligned} M_{1988.5} &= \frac{Y_{1987} + Y_{1988} + Y_{1989} + Y_{1990}}{4} \\ &= \frac{16.74 + 17.12 + 21.84 + 28.48}{4} \\ &= 21.045 \end{aligned}$$

- The 4-point moving average obtained in this way is called a non-central 4-points moving average. In the case of this even number N, the non-central moving average does not match the observation year of the original data, which is inconvenient. In the case of this even number N, it is calculated as the average of the noncentral moving average values of two adjacent non-central moving averages. In other words, the central four-point moving average in 1989 is the average of $M_{1988.5}$ and $M_{1989.5}$ as follows:

$$\begin{aligned} M_{1989} &= \frac{M_{1988.5} + Y_{1989.5}}{2} \\ &= \frac{21.0450 + 21.6475}{2} \\ &= 21.3463 \end{aligned}$$

- If the time series is quarterly or monthly, a 4-point central moving average or a 12-point central moving average is an average of one year, so it is often used to observe data without seasonality.

13.2.2 Exponential Smoothing

- 3-point moving average can be considered the weighted average of three data with each weight $1/3$ as follows:

$$M_t = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3} = 1/3\, Y_{t-1} + 1/3\, Y_t + 1/3\, Y_{t+1}$$

When the weights are $w_1, w_2, \ldots, w_n$, the weighted moving average M_t of the time series is defined as follows:

$$M_t = \sum_{i=1}^{n} w_i Y_i, \quad n \text{ is the number of data.}$$

Here, $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$

- Various weighted averages with different weights can be used depending on the purpose. Among them, a smoothing method that gives more weight to data closer to the present and smaller weights as it is farther from the present is called exponential smoothing. The exponential smoothing method is determined by an exponential smoothing constant α that has a value between 0 and 1. The exponentially smoothed data E_t is calculated as follows:

$$\begin{aligned} E_1 &= \alpha Y_1 + (1-\alpha) E_0 \\ E_2 &= \alpha Y_2 + (1-\alpha) E_1 \\ E_3 &= \alpha Y_3 + (1-\alpha) E_2 \end{aligned}$$

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$$E_t = \alpha Y_t + (1-\alpha)E_{t-1}$$

Here, an initial value E_0 is required, and Y_1 is usually used a lot, and the average value of the data can also be used. The exponentially smoothed value E_t at the point in time t gives weight α to the current data, and the $(1-\alpha)$ weight to the previous smoothed data is given. The exponentially smoothed value E_t can be represented with the original data Y_t as follows:

$$E_t = \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + (1-\alpha)^{t-2} Y_2 + \alpha(1-\alpha)^{t-1} Y_1 + (1-\alpha)^t E_0$$

Therefore, the exponential smoothing method uses all data from the present and the past, but gives the current data the highest weight, and gives a lower weight as the distance from the present time increases.

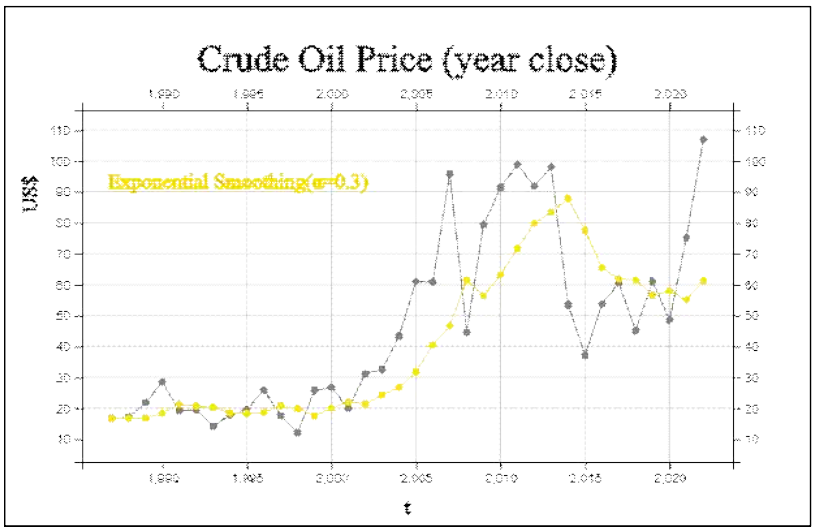
- Exponential smoothing of the crude oil price in [Table 13.2.1] with the initial value $E_{1986} = Y_{1987}$ and exponential smoothing constant α = 0.3 is as follows.

$$E_{1986} = Y_{1987} = 16.74$$
$$E_{1987} = (0.3)Y_{1987} + (1-0.3)E_{1986} = (0.3)(16.74) + (0.7)(16.74) = 16.74$$
$$E_{1988} = (0.3)Y_{1988} + 0.7E_{1987} = (0.3)(17.12) + (0.7)(16.74) = 16.854$$

All data exponentially smoothed with α = 0.3 are given in [Table 13.4]. It can be seen that, in the exponential smoothing method, there is no loss of data at both ends, unlike the moving average method. The crude oil price time series and exponentially smoothed data are shown in <Figure 13.2.2>. It can be seen that the smoothed data are not significantly different from the original data. If the value of α is small, more weight is given to the past data than to the present, making it less sensitive to sudden changes in the present data. Conversely, the closer the value of α is to 1, that is, the more weight is given to the current data, the more the smoothed data resembles the original data, and the smoothing effect disappears.

[Table 13.2.2] Price of Crude Oil and Exponential Smoothing with α=0.3

Year	Price of Oil	α=0.3 Exponental Smoothing	Year	Price of Oil	α=0.3 Exponental Smoothing
1987	16.74	16.740	2005	61.06	40.554
1988	17.12	16.854	2006	60.85	46.643
1989	21.84	18.350	2007	95.95	61.435
1990	28.48	21.389	2008	44.60	56.385
1991	19.15	20.717	2009	79.39	63.286
1992	19.49	20.349	2010	91.38	71.714
1993	14.19	18.501	2011	98.83	79.849
1994	17.77	18.282	2012	91.83	83.443
1995	19.54	18.659	2013	98.17	87.861
1996	25.90	20.832	2014	53.45	77.538
1997	17.65	19.877	2015	37.13	65.416
1998	12.14	17.556	2016	53.75	61.916
1999	25.76	20.017	2017	60.46	61.479
2000	26.72	22.028	2018	45.15	56.580
2001	19.96	21.408	2019	61.14	57.948
2002	31.21	24.348	2020	48.52	55.120
2003	32.51	26.797	2021	75.21	61.146
2004	43.36	31.766	2022	106.95	74.888



<Figure 13.2.2> Price of Crude Oil and Exponential Smoothing with α=0.3

13.2.3 Filtering by Moving Median 필터링

- The **N-point centered moving median** of a time series refers to the median of N data from a single point in time. For example, in crude oil price data, the value of a five-point moving median for a specific year is the median of data for two years before a certain year, that year, and data for two years thereafter. If data are denoted by $Y_{t-2}, Y_{t-1}, Y_t, Y_{t+1}, Y_{t+2}$, and the data are sorted from smallest to largest, and expressed as $Y_{(t-2)}, Y_{(t-1)}, Y_{(t)}, Y_{(t+1)}, Y_{(t+2)}$, the median value is $MovingMedian_t = Y_{(t)}$.
- For example, the 1989 central moving median for crude oil prices in [Table 13.3]

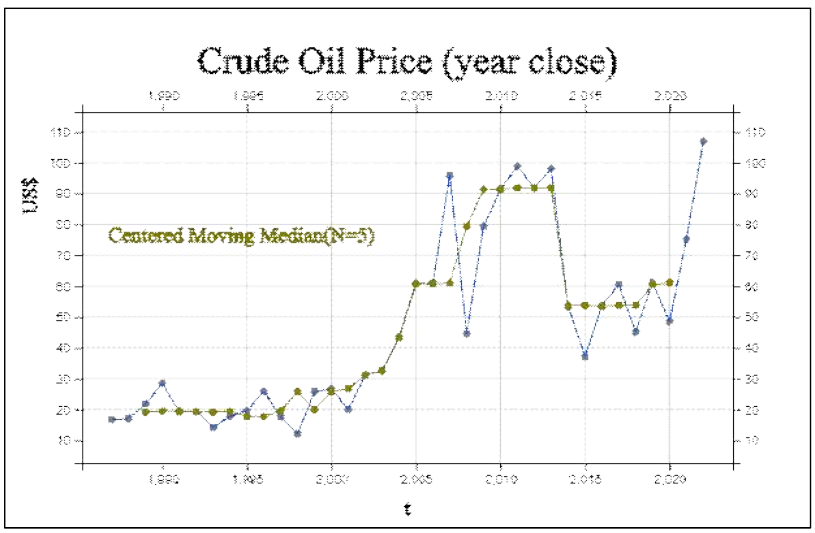
is as follows:

$$\begin{aligned} MovingMedian_{1989} &= median\{Y_{1987}, Y_{1988}, Y_{1989}, Y_{1990}, Y_{1991}\} \\ &= median\{16.74,\ 17.12,\ 21.84,\ 28.48,\ 19.15\} \\ &= 19.15 \end{aligned}$$

- [Table 13.2.3] and <Figure 13.2.3> show all the five-point moving median values obtained in this way and their graphs. Note that the moving median for the first two years and the last two years are not available here. Because the centered moving medians remove extreme values, it is called a filtering and the time series is much smoother than the original data.

[Table 13.2.3] Price of Crude Oil and 5-point Centered Moving Median

Year	Price of Oil	5-point Centered Moving Median	Year	Price of Oil	5-point Centered Moving Median
1987	16.74		2005	61.06	60.85
1988	17.12		2006	60.85	60.85
1989	21.84	19.15	2007	95.95	61.06
1990	28.48	19.49	2008	44.60	79.39
1991	19.15	19.49	2009	79.39	91.38
1992	19.49	19.15	2010	91.38	91.38
1993	14.19	19.15	2011	98.83	91.83
1994	17.77	19.49	2012	91.83	91.83
1995	19.54	17.77	2013	98.17	91.83
1996	25.90	17.77	2014	53.45	53.75
1997	17.65	19.54	2015	37.13	53.75
1998	12.14	25.76	2016	53.75	53.45
1999	25.76	19.96	2017	60.46	53.75
2000	26.72	25.76	2018	45.15	53.75
2001	19.96	26.72	2019	61.14	60.46
2002	31.21	31.21	2020	48.52	61.14
2003	32.51	32.51	2021	75.21	
2004	43.36	43.36	2022	106.95	



<Figure 13.2.3> Price of Crude Oil and 5-point Centered Moving Median

- If the value of N is an even number, there is a difficulty in obtaining the central moving median having the same number of data on both sides of the base year. For example, the center of the four-point moving median from 1987 to 1990 is between 1988 and 1989. If you denote this as $M_{1988.5}$, it can be calculated as follows:

$$\begin{aligned} MovingMedian_{1988.5} &= median\{Y_{1987}, Y_{1988}, Y_{1989}, Y_{1990}\} \\ &= median\{16.74, 17.12, 21.84, 28.48\} \\ &= \frac{17.12+21.84}{2} = 19.48 \end{aligned}$$

The 4-point moving median obtained in this way is called the non-central 4-point moving median. As such, the non-central moving average in the case of this even number N does not match the observation year of the original data, which is inconvenient. In the case of this even number, it is calculated as the average of the values of the two non-central moving averages that are adjacent to each other. In other words, the central four-point moving median in 1989 is the mean of $M_{1988.5}$ and $M_{1989.5}$.

13.3 Transformation of Time Series

- Time series can be viewed by drawing the raw data directly, but in order to examine various characteristics, change in percentage increase or decrease is examined, and an index that is a percentage with respect to base time is alse examined. In addition, in order to examine the relation of the previous data, it is compared with a time lag or converted into horizontal data using the difference. When the variance of the time series increases with time, it is sometimes converted into a form suitable for applying the time series model by using logarithmic, square root, or Box-Cox transformation.

13.3.1 Percent

A. Percentage Change

In a time series, you can examine the increase or decrease of a value, but you can easily observe the change by calculating the percentage increase or decrease. When the time series is expressed as $Y_1, Y_2, \cdots, Y_n$, the percentage increase or decrease P_t compared to the previous data is as follows.

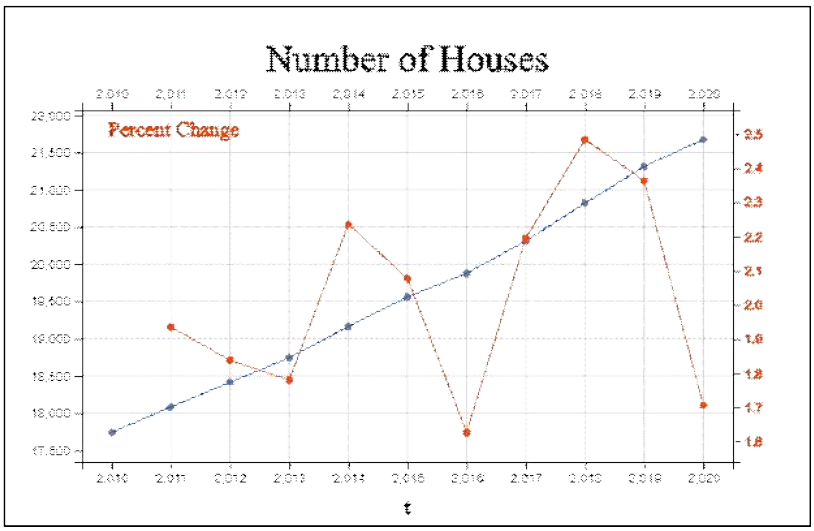
$$P_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}} \times 100,\ t = 2,3,...,n$$

[Table 13.3.1] shows the number of houses in Korea from 2010 to 2020, and <Figure 13.3.1> shows the percentage increase or decrease compared to the previous data. Looking at this rate of change, it can be easily observed that the original time series has an overall increasing trend, but the rate of change of the previous year has many changes. In other words, it can be observed that there was a 2.23% increase in the number of houses in 2014 compared to the previous year, and a 2.48% increase in the number of houses in 2018 as well.

$$P_{2014} = \frac{19161.2 - 18742.1}{18742.1} \times 100 = 2.23$$

[Table 13.3.1] Number of Houses in Korea and Percent Change
(Korea National Statistical Office, unit 1000)

Year	Number of Houses	% Change
2010	17738.8	-
2011	18082.1	1.93
2012	18414.4	1.83
2013	18742.1	1.77
2014	19161.2	2.23
2015	19559.1	2.07
2016	19877.1	1.62
2017	20313.4	2.19
2018	20818.0	2.48
2019	21310.1	2.36
2020	21673.5	1.70



<Figure 13.3.1> Number of Houses in Korea and Percent Change

B. Simple Index

Another way to use percentages to easily characterize changes over time is to calculate an index number. An **index number** is a number that indicates the change over time of a time series. The index number $Index_t$ of a time series at a certain point in time is the percentage of the total time series data for a predetermined time point t_0 called the base period.

$$Index_t = \frac{Y_t}{Y_{t_0}} \times 100, \ t = 1, 2, \ldots, n$$

The most commonly used indices in the economic field are the **price index** and the **quantity index**. For example, the consumer price index is a price index indicating the price change of a set of goods that can reflect the total consumer price, and the index indicating the change in total electricity consumption every year is the quantity index. There are several methods of calculating the index, which are broadly divided into **simple index number** when the number of items

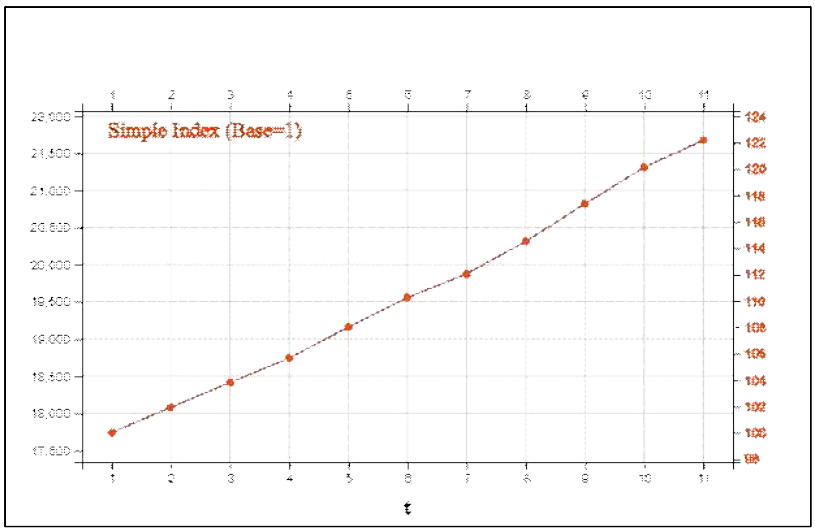
represented by the index is one, and **composite index number** when there are several as in the consumer price index.

[Table 13.3.2] is a simple index for the number of houses in Korea from 2010 to 2020, with the base time being 2010. If you look at the figure for the index, you can see that in this case, there is no significant change from the original time series and trend. It can be seen that there is a 22.18% increase in the number of houses in 2020 compared to 2010.

$$Index_{2020} = \frac{Y_{2020}}{Y_{2010}} \times 100 = \frac{21673.5}{17738.8} \times 100 = 122.18,\ t = 1,2,\ldots,n$$

[Table 13.3.2] Simple Index of Number of Houses in Korea
(Korea National Statistical Office, unit 1000)

Year	Number of Houses	Simple Index Base: 2010
2010	17738.8	100.00
2011	18082.1	101.94
2012	18414.4	103.81
2013	18742.1	105.66
2014	19161.2	108.02
2015	19559.1	110.26
2016	19877.1	112.05
2017	20313.4	114.51
2018	20818.0	117.36
2019	21310.1	120.13
2020	21673.5	122.18



<Figure 13.3.2> Simple Index of Number of Houses in Korea

C. Composite Index

Composite index is a method in which the change in price or quantity of several goods is set at a specific time point as the base period, and then the data at each time point is calculated as a percentage value compared to the base period. An example of the most used composite index is the consumer price index, which reflects price fluctuations of about 500 products in Korea that affect

consumer prices. Other commonly used composite indices include the comprehensive stock index, which examines the price fluctuations of all listed stocks traded in the stock market.

For the composite index, a weighted composite index that is calculated by weighting the price of each product with the quantity consumed is often used. When calculating such a weighted composite index, the case where the quantity consumption at the base time is used as a weight is called the **Laspeyres method,** and the case where the quantity consumption at the current time is used as the weight is called the **Paasche method.** In general, the Laspeyres method of weighted composite index is widely used, and the consumer price index is a representative example. The price index of the Paasche method is used when the consumption of goods used as weights varies greatly over time, and can be used only when the consumption at each time point is known. It is expensive to examine the quantity consumption at each point in time.

Assuming that $P_{1t}, \cdots, P_{kt}$ are the prices of k number of products at the time point t, and $Q_{1t_0}, \cdots, Q_{kt_0}$ are the quantities of each product consumpted at the base time, the formula for calculating each composite index is as follows:

Laspeyres Index: $$Index_t = \frac{Q_{1t_0}P_{1t} + \cdots + Q_{kt_0}P_{kt}}{Q_{1t_0}P_{1t_0} + \cdots + Q_{kt_0}P_{kt_0}} \times 100$$

Paasche Index: $$Index_t = \frac{Q_{1t}P_{1t} + \cdots + Q_{kt}P_{kt}}{Q_{1t}P_{1t_0} + \cdots + Q_{kt}P_{kt_0}} \times 100$$

The data in [Table 13.3.3] shows the price and quantity of three metals by month in 2020.

[Table 13.3.3] Composite Index of three Metal Prices($/ton) and Production Quantity(ton)

Month	Copper		Metal		Lead		Laspeyres	Paasche
	Price	Quantity	Price	Quantity	Price	Quantity		
1	1361.6	100.7	213	4311	530.0	46.1	100.00	100.00
2	1399.0	95.1	213	4497	520.0	47.0	100.31	100.28
3	1483.6	104.0	213	5083	529.0	51.0	101.13	101.01
4	1531.6	95.6	213	5077	540.0	23.0	101.63	101.35
5	1431.2	103.3	213	5166	531.0	26.5	100.65	100.57
6	1383.8	106.9	213	4565	580.0	13.5	100.42	100.27
7	1326.8	95.9	213	4329	642.8	27.4	100.16	99.98
8	1328.8	96.7	213	4057	602.6	25.8	100.00	99.87
9	1307.8	95.7	213	3473	513.6	20.5	99.43	99.38
10	1278.4	89.1	213	3739	480.8	24.6	99.01	99.07
11	1354.2	100.5	213	3817	528.4	21.5	99.92	99.92
12	1305.2	96.9	213	3694	462.2	27.9	99.18	99.21

In [Table 13.3.3], the Laspeyres index for the data for February with January as the base time is as follows.

$$Index_t = \frac{Q_{1t0}P_{1t} + \cdots + Q_{kt0}P_{kt}}{Q_{1t0}P_{1t0} + \cdots + Q_{kt0}P_{kt0}} \times 100$$
$$= \frac{(100.7)(1399.0) + (4311)(213) + (46.1)(520)}{(100.7)(1361.6) + (4311)(213) + (47.0)(530)} = 100.31$$

Similarly, Paasche index is as follows:

$$Index_t = \frac{Q_{1t}P_{1t} + \cdots + Q_{kt}P_{kt}}{Q_{1t}P_{1t0} + \cdots + Q_{kt}P_{kt0}} \times 100$$
$$= \frac{(95.1)(1399.0) + (4497)(213) + (47.0)(520)}{(95.1)(1361.6) + (4497)(213) + (47.0)(530)} = 100.28$$

In [Table 13.3.3], it can be seen that the production quantity of iron and lead in the last 4 quarters is significantly different from the production quantity in January, which is the base time. In this way, when the quantity fluctuates greatly and the quantity at each time is known, the Pasche index can be said to be the best index because it appropriately reflects the price change at that time.

13.3.2 Time Lag and Difference

A. Time Lag

In a time series, current data can usually be related to past data. Lag means a transformation for comparing data of the present time and observation values at one time point or a certain past time point. That is, when the observed time series is $Y_1, Y_2, \cdots, Y_n$, the time series with lag 1 becomes $-, Y_1, Y_2, \cdots, Y_{n-1}$. Note that, in case of lag k, there are no data for the first k number than the original data.

The correlation coefficient between the time lag data and the raw data is called the **autocorrelation coefficient**. If the average of time series is $\overline{Y}$, the k -lag autocorrelation r_k is defined as follows:

$$r_k = \frac{\sum_{t=k+1}^{n} (Y_t - \overline{Y})(Y_{t-k} - \overline{Y})}{\sum_{t=1}^{n} (Y_t - \overline{Y})^2} \quad k=0,1,2,\cdots,n-1$$

$r_1, r_2, \cdots, r_k$ are called an autocorrelation function and are used to determine a time series model.

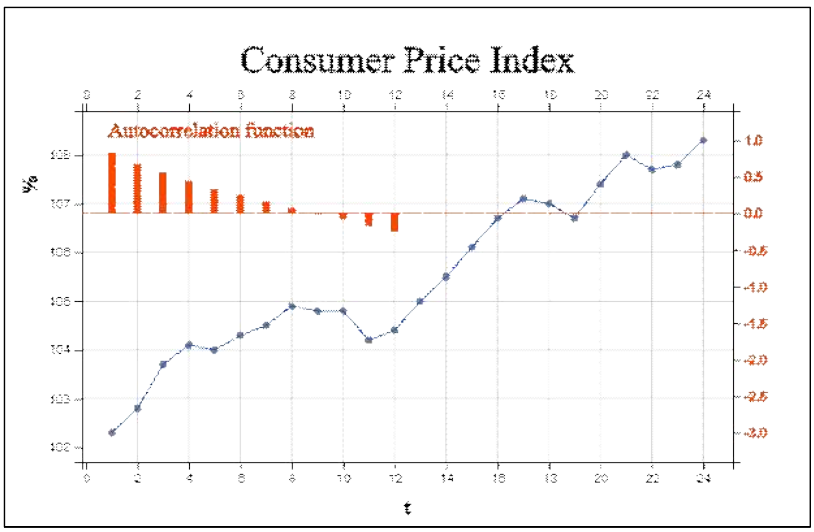
[Table 13.3.4] shows the monthly consumer price index for the past two years and time lag 1 to 12 for this data, and the autocorrelation coefficients are shown in [Table 13.3.5]. <Figure 13.3.3> shows the original time series and the autocorrelation function.

[Table 13.3.4] Monthly Consumer Price Index and Time Lag 1, Lag 2, ... Lag 12

time	Year/month	CPI	Lag 1	Lag 2	...	Lag 12
1	2020.01	102.3	-	-		-
2	2020.02	102.8	102.3	-		-
3	2020.03	103.7	102.8	102.3		-
4	2020.04	104.1	103.7	102.8		-
5	2020.05	104.0	104.1	103.7		-
6	2020.06	104.3	104.0	104.1		-
7	2020.07	104.5	104.3	104.0		-
8	2020.08	104.9	104.5	104.3		-
9	2020.09	104.8	104.9	104.5		-
10	2020.10	104.8	104.8	104.9		-
11	2020.11	104.2	104.8	104.8		-
12	2020.12	104.4	104.2	104.8		-
13	2021.01	105.0	104.4	104.2	...	102.3
14	2021.02	105.5	105.0	104.4		102.8
15	2021.03	106.1	105.5	105.0		103.7
16	2021.04	106.7	106.1	105.5		104.1
17	2021.05	107.1	106.7	106.1		104.0
18	2021.06	107.0	107.1	106.7		104.3
19	2021.07	106.7	107.0	107.1		104.5
20	2021.08	107.4	106.7	107.0		104.9
21	2021.09	108.0	107.4	106.7		104.8
22	2021.10	107.7	108.0	107.4		104.8
23	2021.11	107.8	107.7	108.0		104.2
24	2021.12	108.3	107.8	107.7		104.4

[Table 13.3.5]
Autocorrelation Function

t	autocorrelation
1	0.8318
2	0.6772
3	0.5651
4	0.4479
5	0.3333
6	0.2547
7	0.1647
8	0.0755
9	-0.0143
10	-0.0854
11	-0.1737



<Figure 13.3.3> Autocorrelation Function Graph

B. Differencing

Since the price index in [Table 13.3.4] has a linear trend, a model for this trend can be built, but in some cases, a model can be created by changing the time series to a horizontal trend. The way to transform a linear trend into a horizontal trend is to use a differencing. When the time series is $Y_1, Y_2, \cdots, Y_n$, the first order difference ∇Y_t is as follows:

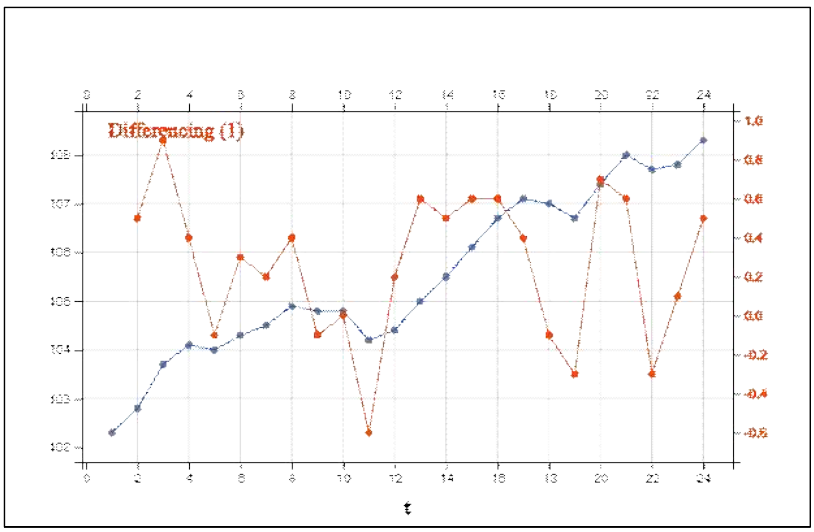
$$\nabla Y_t = Y_t - Y_{t-1} \quad , \quad t=2,3,\cdots,n$$

If the raw data is a linear trend, the first-order differencing of time series is a horizontal time series because it means a change in slope. If we make differencing on the first-order differencing ∇Y_t, it becomes the second-order difference as follows:

$$\nabla^2 Y_t = \nabla Y_t - \nabla Y_{t-1} = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}), \ t = 3,4,\cdots,n$$

If the raw data has a trend with a quadratic curve, the second differencing of time series becomes a horizontal time series.

<Figure 13.3.4> shows the first order differencing of [Table 13.3.4] time series and it becomes horizontal series.



<Figure 13.3.4> 1st Order Differencing of Consumer Price Index

13.3.3 Mathematical Transformation

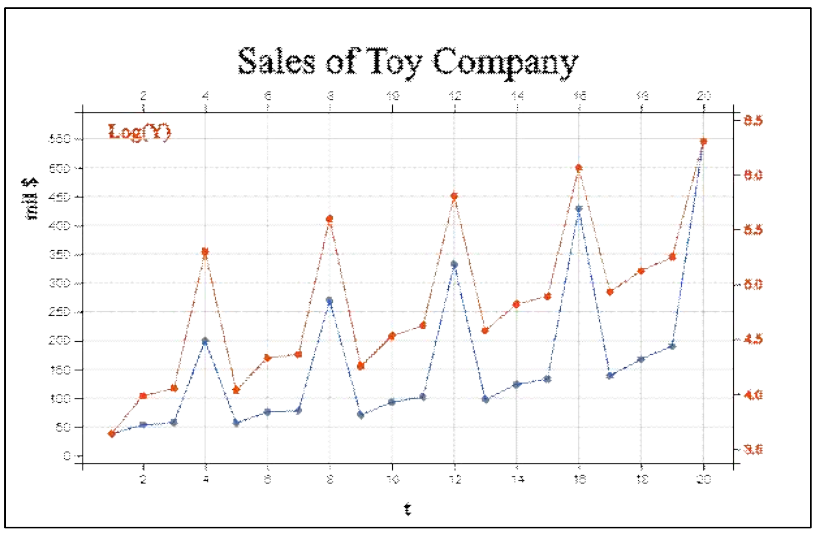
- If the original data of the time series is used as it is, modeling may not be easy or it may not satisfy various assumptions. In this case, we can fit the model we want by performing an appropriate functional transformation, such as log transformation. The functions commonly used for mathematical transformations are as follows.

Log function	$W = \log(Y)$
Square root function	$W = \sqrt{Y}$
Square function	$W = Y^2$
Box-Cox Transformation	$W = \begin{cases} \dfrac{Y^p - 1}{p}, & p \neq 0 \\ \log(Y), & p = 0 \end{cases}$

- [Table 13.3.6] is a toy company's quarterly sales, and <Figure 13.3.5> is a diagram of this data. It is a seasonal data by quarter, but, as time goes on, dispersion of sales increases over time. It is not easy to apply a time series model to data with this increasing dispersion over time. In this case, log transformation $W = \log(Y)$ can reduce the dispersion as time increases, as shown in <Figure 13.3.5>, so that a model can be applied. After predicting by applying the model to log-transformed data, exponential transformation $Y = \exp(W)$ is performed again to predict the raw data.

[Table 13.3.6] Quarterly Sales of a Toy Company (unit million $)

t	Year	Sales
1	2017 Quarter 1	38.0
2	Quarter 2	53.6
3	Quarter 3	57.5
4	Quarter 4	200.0
5	2018 Quarter 1	56.5
6	Quarter 2	75.8
7	Quarter 3	78.3
8	Quarter 4	269.7
9	2019 Quarter 1	70.2
10	Quarter 2	92.7
11	Quarter 3	101.8
12	Quarter 4	332.6
13	2020 Quarter 1	97.3
14	Quarter 2	123.7
15	Quarter 3	132.9
16	Quarter 4	429.4
17	2021 Quarter 1	138.3
18	Quarter 2	167.6
19	Quarter 3	189.9
20	Quarter 4	545.9



<Figure 13.3.5> Log Transformation of Toy Sales

- The square root transform is used for a similar purpose to the log transform, and the square transform can be used when the variance decreases with time. The Box-Cox transform is a general transformation.

13.4 Regression Model and Forecasting

- If there is a trend factor that shows a continuous increase or decrease in the time series, the regression model learned in Chapter 12 can be applied. For example, if the time series shows a linear trend, the linear regression model is applied with the time series as the observation values of the random variable $Y_1, Y_2, \ldots, Y_n$ and time as 1, 2, … , n as follows.

$$Y_t = \beta_0 + \beta_1 \cdot t + \epsilon_t$$

Here ε_t is the error term with mean 0 and variance σ^2.

A characteristic of the linear model is that it increases by a slope β_1 of a certain magnitude over time.

- When the estimated regression coefficients are $\hat{\beta}_0, \hat{\beta}_1$, the validity test of the linear regression model is the same as the method described in Chapter 12. The standard error of estimate and coefficient of determination are often used. In a linear trend model, σ represents the degree to which observations can be scattered around the estimated regression line at each time point. As an estimate of this σ, the following standard error is used.

$$s = \sqrt{\frac{1}{n-2}\sum_{t=1}^{n}(Y_t - \hat{Y}_t)^2}$$

A smaller standard error value s indicates that the observed values are close to the estimated regression line, which means that the regression line model is well fitted.

- The coefficient of determination is the ratio of the regression sum of squares, RSS, which is explained out of the total sum of squares, TSS.

$$R^2 = \frac{\mathrm{RSS}}{\mathrm{TSS}}$$

The value of the coefficient of determination is always between 0 and 1, and the closer the value is to 1, the more dense the samples are around the regression line, which means that the estimated regression equation explains the observations well.

- As explained in Chapter 12, since it is difficult to determine the absolute criteria for adequacy of the standard error or the coefficient of determination, a hypothesis test is used to determine whether the trend parameter β_1 is zero or not.

Hypothesis: $\mathrm{H}_0 : \beta_1 = 0$, $\mathrm{H}_1 : \beta_1 \neq 0$

Test statistic: $t_{obs} = \dfrac{\hat{\beta}_1}{\mathrm{SE}(\hat{\beta}_1)}$, Here $\mathrm{SE}(\hat{\beta}_1) = \dfrac{s}{\sqrt{\sum_{t=1}^{n}(t-\bar{t})^2}}$

Rejection region: If $|t_{obs}| > t_{n-2,\alpha/2}$, reject H_0 with significance level α.

If the null hypothesis $\mathrm{H}_0 : \beta_1 = 0$ is not rejected, the model cannot be considered valid.

- The assumption for error ϵ_t is tested using the residual, which is the difference between the observed time series value and the predicted value which is called residual analysis. Residual analysis usually examines whether assumptions about error terms such as independence and equal variance between errors are satisfied by drawing a scatter plot of the residuals over time or a scatter plot of the residuals and predicted values. In the scatterplots, if the residuals do not show a specific trend around 0 and appear randomly, it means that each assumption is valid. To examine the normality assumption of the error term, draw a normal probability plot of the residuals, and if the points on the figure show the shape of a straight line, it is judged that the assumption of the normal distribution is appropriate.

- If the linear regression model is suitable, the predicted value $\hat{Y}_{t_0} = \hat{\beta}_0 + \hat{\beta}_1 \cdot t_0$ at the time point t_0 can be interpreted as a point estimate for the mean of the random variable Y_{t_0} at the time point, and the confidence interval for the mean of $\hat{Y}_{t_0}$ at this time t_0 is as follows:

$$\hat{Y}_{t_0} \pm t_{n-2,\alpha/2} \cdot \mathrm{SE}(\hat{Y}_{t_0})$$

Here $\mathrm{SE}(\hat{Y}_{t_0}) = s \cdot \sqrt{\frac{1}{n} + \frac{(t_0 - \bar{t})^2}{\sum_{i=1}^{n}(i - \bar{t})^2}}$

- If the trend is in the form of a quadratic, cubic or higher polynomial, the following multiple linear regression model can be assumed.

(Quadratic) $Y_t = \beta_0 + \beta_1 \cdot t + \beta_2 \cdot t^2 + \epsilon_t$

(Cubic) $Y_t = \beta_0 + \beta_1 \cdot t + \beta_2 \cdot t^2 + \beta_3 \cdot t^3 + \epsilon_t$

The prediction method is similar to the above simple linear regression model.

- If the trend is not a polynomial model as above, the following model can also be considered.

(Square root) $Y_t = \beta_0 + \beta_1 \cdot \sqrt{t} + \epsilon_t$

(Log) $Y_t = \beta_0 + \beta_1 \cdot \log(t) + \epsilon_t$

These models are the same as the linear regression model if I $\sqrt{t}$ or $\log(t)$ are replaced with a variable X in the simple linear regression and the prediction method is similar.

- In addition, the function types to which the linear regression model can be applied by transformation are as follows.

(Power) $Y_t = \beta_0 \cdot t^{\beta_1} + \epsilon_t$

(Exponential) $Y_t = \beta_0 \cdot e^{(\beta_1 t)} + \epsilon_t$

In the case of these two models, the parameters should be estimated using the nonlinear regression model, but if the error term is ignored, the linear model can be estimated approximately as follows:

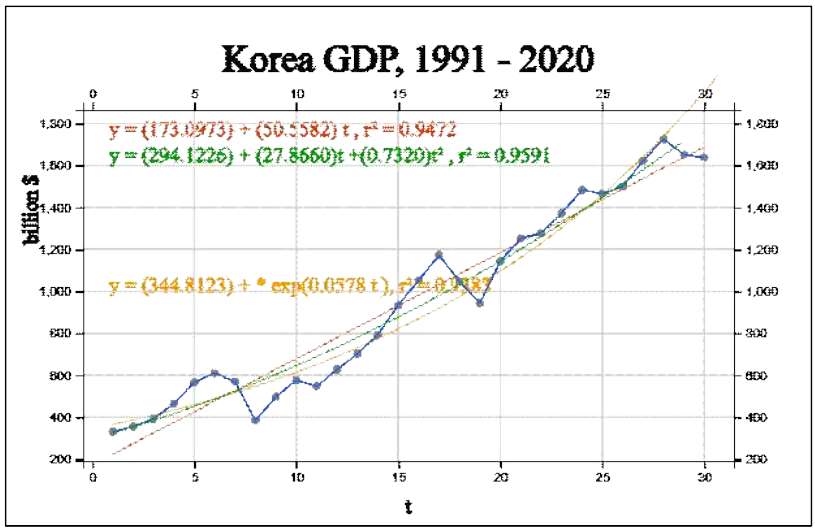
(Power) $\log(Y_t) = \log(\beta_0) + \beta_1 \cdot \log(t)$

(Exponential) $\log(Y_t) = \log(\beta_0) + \beta_1 t$

- Korea's GDP from 1986 to 2021 is shown in [Table 13.4.1]. <Figure 13.4.1> shows the application of three regression models to this data. Among these models, the quadratic model has the largest value of r^2 = 0.9591, so it can be said that the time series is the most suitable model. However, additional validation of the model is required.

[Table 13.4.1] GDP of Korea

Year	GDP (billion $)
1991	330.65
1992	355.53
1993	392.67
1994	463.62
1995	566.58
1996	610.17
1997	569.76
1998	383.33
1999	497.51
2000	576.18
2001	547.66
2002	627.25
2003	702.72
2004	793.18
2005	934.9
2006	1053.22
2007	1172.61
2008	1047.34
2009	943.67
2010	1143.98
2011	1253.16
2012	1278.43
2013	1370.8
2014	1484.32
2015	1465.77
2016	1499.36
2017	1623.07
2018	1725.37
2019	1651.42
2020	1638.26



<Figure 13.4.1> GDP of Korea and Three Regression Model

13.5 Exponential Smoothing Model and Forecasting

- When the time series moves in a trend, the future can be predicted well with the regression model. However, it may not be appropriate to predict a time series that is dynamically moving hourly, daily, etc. In this case, a moving average model or an exponential smoothing model can be used. The time series model is explained into two cases where the trend is stationary and linear.

13.5.1 Stationary Time Series

- A time series is called stationary if the statistical properties such as mean, variance and covariance are consistent over time. When a time series is the observed values of random variables $Y_1, Y_2, \cdots, Y_T$, a stationary time series is the following model that changes around a constant value μ.

$$Y_i = \mu + \epsilon_i \ , \ i = 1, 2, ..., T$$

Here μ is unknown parameter and ϵ_i is an error term which is independent with mean 0 and variance σ^2.

A. Single Moving Average Model

In a stationary time series model, the estimated value of μ, $\hat{\mu}$, is the mean of the data. .

$$\hat{\mu} = \frac{1}{T}\sum_{i=1}^{T} Y_i$$

Using this model, the prediction after τ time points ahead at the current time T denoted as $\hat{Y}_{T+\tau}$, is as follows:

$$\hat{Y}_{T+\tau} = \hat{\mu} \ , \ \tau = 1, 2, \cdots$$

It is called a **simple average model.** .

The simple average model uses all observations until the current time. However, the unknown parameter μ may shift slightly over time, so it would be reasonable to give more weight to recent data than to past data for prediction. If a weight $\frac{1}{N}$ is given to only the most recent N observations at the present time T and the weight of the remaining observations is set to 0, the estimated value of μ is as follows.

$$\hat{\mu} = \frac{1}{N}\sum_{i=T-N+1}^{T} Y_i = \frac{1}{N}\left(Y_{T-N+1} + Y_{T-N+2} + \cdots + Y_T \right)$$

This is called a s**ingle moving average model** at the time point T and is denoted by M_T. The N single moving average means the average of the observations adjacent to the time point T. Notice that $Y_1, Y_2, \ldots, Y_T$ are independent of each other by assumption, but $M_1, M_2 \ldots\ldots M_T$ are not independent of each other, but are correlated.

The value of the single moving average varies depending on the size of N. When the value of N is large, it becomes insensitive to the fluctuations of the original time series, so it changes gradually, and when the value of N is small, it becomes sensitive to fluctuations. Therefore, when the fluctuation of the original

time series is small, it is common to set the small value of N, and when the fluctuation is large, it is common to set the value of large N.

Using the single moving average model at the time point T, the predicted value and the mean and variance of the predicted value at the time point $T+\tau$ are as follows.

$$\hat{Y}_{T+\tau} = M_T \ , \ \tau = 1, 2, \cdots$$
$$E(\hat{Y}_{T+\tau}) = E(M_T) = \mu$$
$$Var(\hat{Y}_{T+\tau}) = Var(M_t) = \frac{\sigma^2}{N}$$

When the single moving average model is used, the 95% confidence interval estimation of the predicted value is approximately as follows.

$$\hat{Y}_{T+\tau} \pm 1.96\sqrt{Var(\hat{Y}_{T+\tau})}$$
$$\Rightarrow M_T \pm 1.96\sqrt{\frac{MSE}{N}}$$

The monthly sales for the last two years of a furniture company are as shown in [Table 13.4.2], and the residual between the raw data and the predicted value of one point in time was calculated by obtaining a six-point moving average. <Figure 13.5.1> shows the time series for this. This time series fluctuates up and down based on approximately 95, and such a time series is called a stationary time series.

When N = 6, the moving average for the first 5 time points cannot be obtained. The moving average at time 6 is is as follows:

$$M_6 = \frac{95+100+87+123+90+96}{6} = 98.50$$

Therefore, one time prediction at time 6 becomes $\hat{Y}_{6+1} = 98.50$ and the residual at time 7 is as follows:

$$e_7 = Y_7 - \hat{Y}_{6+1} = 75 - 98.50 = -23.50$$

In the same way, the moving average of the remaining time points, the predicted values after one time point, and the residuals are as shown in [Table 13.5.1], so the mean square error is as follows:

$$MSE = \frac{\sum_{i=7}^{24}\left(Y_i - \hat{Y}_i\right)^2}{18} = 331.22$$

Sales for the next three months are the last moving average M_{24}, and the 95% confidence interval for the forecast is as follows:

$$\hat{Y}_{24+1} = \hat{Y}_{24+2} = \hat{Y}_{24+3} = M_{24} = 104.67$$
$$M_T \pm 1.96\sqrt{\frac{MSE}{N}}$$
$$\Rightarrow 104.67 \pm 1.96\sqrt{\frac{331.22}{6}}$$
$$\Rightarrow [90.10,\ 119.23]$$

※ **Moving average at initial period**

Since the N-point single moving average cannot be obtained before the time

point N, the prediction model cannot be applied. When there are many time series, this may not be a big problem, but when the number of data is small, it can affect the prediction. In order to solve this problem, the moving average at initial period can be obtained as follows until the time point $N-1$.

$$t=1, \qquad M_1 = Y_1$$

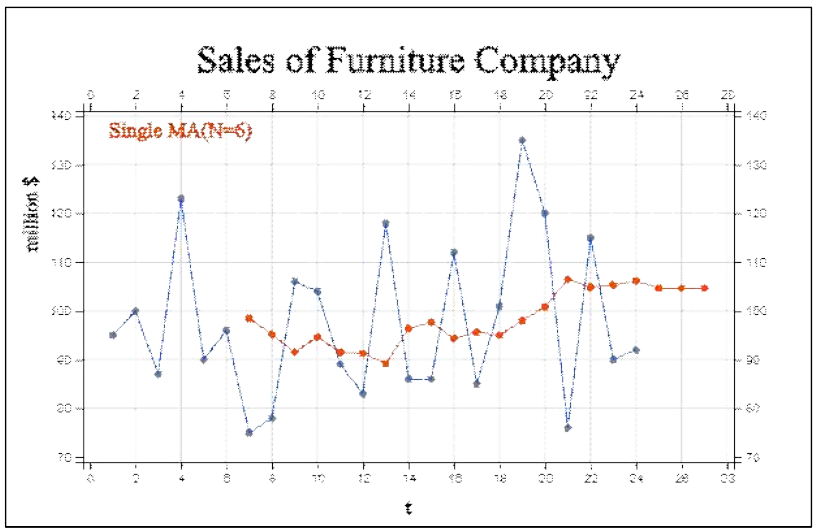
$$t=2, \qquad M_2 = \frac{Y_1 + Y_2}{2}$$

$$\cdots$$

$$t=N-1, \quad M_{N-1} = \frac{Y_1 + Y_2 + \cdots + Y_{N-1}}{N-1}$$

[Table 13.5.1] Montly Sales of a Furniture Company and 6-point Moving Average, One Time Forecast and Residuals

Time t	Sales (unit million $) Y_t	6-pt Moving Average M_t	One Time Forecast $\hat{Y}_t$	Residual $e_t = Y_t - \hat{Y}_t$
1	95			
2	100			
3	87			
4	123			
5	90			
6	96	98.50		
7	75	95.17	98.50	-23.50
8	78	91.50	95.17	-17.17
9	106	94.67	91.50	14.50
10	104	91.50	94.67	9.33
11	89	91.33	91.50	-2.50
12	83	89.17	91.33	-8.33
13	118	96.34	89.17	28.83
14	86	97.67	96.34	-10.33
15	86	94.34	97.67	-11.67
16	112	95.67	94.34	17.67
17	85	95.00	95.67	-10.67
18	101	98.00	95.00	6.00
19	135	100.84	98.00	37.00
20	120	106.50	100.84	19.17
21	76	104.84	106.50	-30.50
22	115	105.34	104.84	10.17
23	90	106.17	105.34	-15.33
24	92	104.67	106.17	-14.17



<Figure 13.5.1> Montly Sales of a Furniture Company and 6-point Moving Average and One Time Forecast

B. Single Exponential Smoothing Model

In the single moving average model, the same weight $\frac{1}{N}$ is given to only the latest N observations, and the previous observations are completely ignored by setting the weight to 0. The **single exponential smoothing method** compensates for the shortcomings of the moving average model by assigning weights to all observations when predicting future values from past observations, but giving more weight to recent data. This single exponential smoothing model uses the value of the exponential smoothing method as the predicted value.

The single exponential smoothing model calculates the weighted average of the exponential smoothing estimator $\hat{\mu}_t$ at the immediately preceding time point and the observation value Y_t at the time point t. Assuming that the exponential smoothing estimated value at the time point t is $S_t = \hat{\mu}_t$ and α is a real number between 0 and 1, the single exponential smoothing value S_t is defined as follows.

$$
\begin{aligned}
S_1 &= \alpha\ Y_1 + (1-\alpha)S_0, \\
S_2 &= \alpha\ Y_2 + (1-\alpha)S_1, \\
&\cdots \\
S_t &= \alpha\ Y_t + (1-\alpha)S_{t-1},
\end{aligned}
$$

Here, α is called the smoothing constant, and the single exponential smoothing value is the weighted average value given the weight α of the most recent observation and $(1-\alpha)$ of the exponential smoothing value S_{t-1} at the time $t-1$. You can better understand the meaning of exponential smoothing if you write down the recursive equation as follows:

$$
\begin{aligned}
S_t &= \alpha Y_t + (1-\alpha)S_{t-1} \\
&= \alpha\ Y_t + (1-\alpha)(\alpha Y_{t-1} + (1-\alpha)S_{t-2}) \\
&= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + (1-\alpha)^2 S_{t-2} \\
&= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \ldots + \\
&\alpha(1-\alpha)^{(t-1)} Y_1 + (1-\alpha)^t S_0
\end{aligned}
$$

In other words, for the single exponential smoothing value S_t, the most recent observation Y_t is given a weight α, and the next most recent observation is given $\alpha(1-\alpha)$, the next is $\alpha(1-\alpha)^2, \cdots$ and so on, a gradually smaller weight. Therefore, if the size of α is small, the current observation value is given a small weight, and the exponential smoothing value is insensitive to the fluctuations of the time series. if the size of α is large, the current observation value is given a large weight, and the exponential smoothing value is sensitive to the fluctuations of the time series. In general, a value between 0.1 and 0.3 is often used as the value of α.

In order to obtain a single exponential smoothing value, an initial smoothing value S_0 is required, and the first observation value or the sample average of several initial data or the overall sample average can be used. The exponential smoothing method has the advantage of being less affected by extreme point or intervention than the ARIMA model and easy to use, although the selection of the smoothing constant is arbitrary and it is difficult to obtain a prediction interval.

The predicted value, average and variance of the predicted value at the time point $T+\tau$ using the single exponential smoothing model are as follows:

$$\begin{aligned}
\hat{Y}_{T+\tau} &= S_T \\
E(\hat{Y}_{T+\tau}) &= E(S_T) = \mu \\
Var(\hat{Y}_{T+\tau}) &= Var(S_T) = \frac{\alpha}{2-\alpha}\sigma^2
\end{aligned}$$

Therefore, when the single exponential smoothing model is used, the 95% interval estimation in the predicted value is approximately as follows.

$$\begin{aligned}
&\hat{Y}_{T+\tau} \pm 1.96\sqrt{Var(\hat{Y}_{T+\tau})} \\
&\Rightarrow S_T \pm 1.96\sqrt{\frac{\alpha}{2-\alpha}MSE}
\end{aligned}$$

To the data of [Table 13.5.1], predict sales for the next three months by a single exponential smoothing model with smoothing constant α = 0.1. Let's use the first observed value for the initial value of exponential smoothing, that is S_0 = Y_1 = 95. The exponential smoothing value for the first three time series are as follows:

$$\begin{aligned}
S_1 &= 0.1 \times Y_1 + (1-0.1) \times S_0 = 0.1 \times 95 + 0.9 \times 95 = 95 \\
S_2 &= 0.1 \times Y_2 + (1-0.1) \times S_1 = 0.1 \times 100 + 0.9 \times 95 = 95.50 \\
S_3 &= 0.1 \times Y_3 + (1-0.1) \times S_2 = 0.1 \times 87 + 0.9 \times 95.5 = 94.65
\end{aligned}$$

At each time point, the prediction after one point in time is as follows:

$$\begin{aligned}
\hat{Y}_{0+1} &= S_0 = 95.00 \\
\hat{Y}_{1+1} &= S_1 = 95.00 \\
\hat{X}_{2+1} &= S_2 = 95.50
\end{aligned}$$

Hence the residuals using the above estimated values are as follows:

$$\begin{aligned}
e_1 &= Y_1 - \hat{Y}_{0+1} = 95.00 - 95.00 = 0 \\
e_2 &= Y_2 - \hat{Y}_{1+1} = 100.00 - 95.00 = 5.00 \\
e_3 &= Y_3 - \hat{Y}_{2+1} = 87.00 - 95.50 = -8.50
\end{aligned}$$

In the same way, the single exponential smoothing of the remaining time points, the predicted values after one time point, and the residuals are as shown in [Table 13.4.2]. Therefore, the mean square error is as follows:

$$\mathrm{MSE} = \frac{\sum_{i=1}^{24}\left(Y_i - \hat{Y}_i\right)^2}{24} = 269.72$$

In terms of mean square error, the MSE of the 6-point single moving average model is 331.22, so it can be said that the exponential smoothing model has better fit.

Sales for the next three months are the last moving average S_{24}, and the 95% confidence interval for the forecast is as follows:

$$\hat{Y}_{24+1} = \hat{Y}_{24+2} = \hat{Y}_{24+3} = S_{24} = 98.66$$

$$S_T \pm 1.96\sqrt{\frac{\alpha}{2-\alpha}MSE}$$

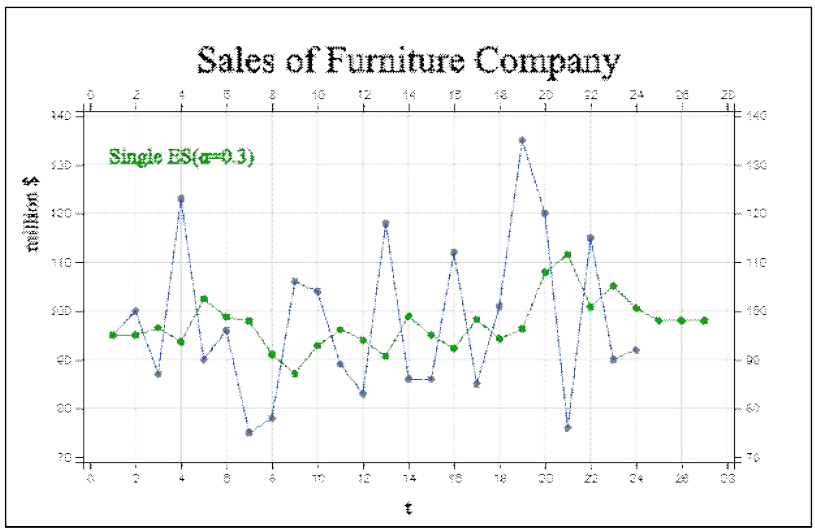
=> $98.66 \pm 1.96\sqrt{\frac{0.1}{2-0.1}269.72}$

=> [65.27, 132.05]

[Table 13.5.2] summarizes the above equations, and <Figure 13.5.2> shows the prediction after one time point and the prediction for the next 3 months using the single exponential smoothing model with α = 0.1.

[Table 13.5.2] Exponential Smoothing with α = 0.1, One Time Forecast and Residual

Time t	Sales (unit million $) Y_t	Exponential Smoothing S_t	One Time Forecast $\hat{Y}_t$	Residual $e_t = Y_t - \hat{Y}_t$
1	95	95.00	95.00	0.00
2	100	95.50	95.00	5.00
3	87	94.65	95.50	-8.50
4	123	97.48	94.65	28.35
5	90	96.74	97.48	-7.48
6	96	96.66	96.74	-0.74
7	75	94.50	96.66	-21.66
8	78	92.85	94.50	-16.50
9	106	94.16	92.85	13.15
10	104	95.15	94.16	9.84
11	89	94.53	95.15	-6.15
12	83	93.38	94.53	-11.53
13	118	95.84	93.38	24.62
14	86	94.86	95.84	-9.84
15	86	93.97	94.86	-8.86
16	112	95.77	93.97	18.03
17	85	94.70	95.77	-10.77
18	101	95.33	94.70	6.30
19	135	99.29	95.33	39.67
20	120	101.36	99.29	20.71
21	76	98.83	101.36	-25.36
22	115	100.45	98.83	16.17
23	90	99.40	100.45	-10.45
24	92	98.66	99.40	-7.40



<Figure 13.5.2> Exponential Smoothing with α = 0.1 and One Time Forecast

※ **Initial value of exponential smoothing**

Since the initial exponential smoothing value S_0at the time point $t=1$ cannot be obtained, the following three methods are commonly used.

1) The first observation, i.e. $S_0 = Y_1$,
2) Partial average using the initial n observation values, that is
$$S_0 = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n),$$
3) The mean up to the entire time point T, i.e.,
$$S_0 = \frac{1}{T}(Y_1 + Y_2 + \cdots + Y_T)$$

※ **Initial smoothing constant**

The same smoothing constant α can be applied to all time series, but the following method is also used to reduce the effect of the initial value S_0.

$$\alpha_t = \frac{1}{t}$$, until α_t reaches α

13.5.2 Linear Trend Time Series

A. Double Moving Average Model

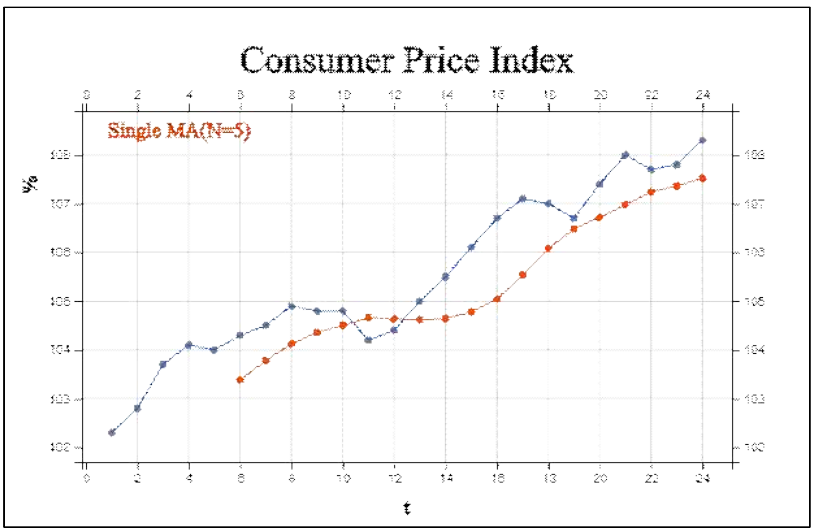
In the previous section, we examined that the single moving average model can be applied to a stationary time series. What would happen if the single moving average model was applied to a time series with a linear trend? That is, for a time series with a linear trend $Y_t = \beta_0 + \beta_1 \cdot t + \epsilon_t$, the N-point single moving average at time T is as follows:

$$M_T = \frac{1}{N}(Y_{T-N+1} + Y_{T-N+2} + \cdots + Y_T)$$

The expected value can be shown as follows:

$$E(M_T) = \beta_0 + \beta_1 T - \frac{N-1}{2}\beta_1$$

That is, in the case of a linear trend model, it can be seen that the single moving average M_T is biased by $\frac{N-1}{2}\beta_1$. For example, if the consumer price index with a linear trend in [Table 13.5.3] is predicted after one point in time using the 5-point single moving average, it is as shown in <Figure 13.5.3>. It can be seen that the predicted value using M_5 is under estimated value of the time series .



<Figure 13.5.3> 5-pt Moving Average of Consumer Price Index with Linear Trend

In the case of a linear trend, one way to eliminate the bias of the single moving average model is the **double moving average**, which obtains the moving average again for the single moving average. The N-point double moving average $M_T^{(2)}$ at the time T and its expected value are as follows.

$$M_T^{(2)} = \frac{1}{N}(M_T + M_{T-1} + \cdots + M_{T-N+1})$$
$$E(M_T^{(2)}) = \beta_0 + \beta_1 T - (N-1)\beta_1$$

Since $E(M_T)$ and $E(M_T^{(2)})$ have the same number of parameters, β_0, β_1 can be estimated by solving the system of two equations as follows:

$$\hat{\beta}_1 = \frac{2}{N-1}(M_T - M_T^{(2)})$$
$$\hat{\beta}_0 = 2M_T - M_T^{(2)} - \hat{\beta}_1 T$$

Therefore, the predicted value at the time point $T+\tau$ using the double moving average at time T as follows.

$$\hat{Y}_{T+\tau} = 2M_T - M_T^{(2)} + \tau\left(\frac{2}{N-1}\right)(M_T - M_T^{(2)})$$

Such a double moving average model can be said to be a kind of heuristic

method. That is, although logical, it is not based on any optimization such as least squares method. However, it can be approximated by the least-squares method, which we will omit in this book.

[Table 13.5.3] is a calculation table for predicting the consumer price index using the 5-point double moving average model. Note that the third column is a 5-point single moving average M_t, but the single moving average cannot be calculated from time points 1 to 4. The fourth column is the calculation of the 5-point double moving average $M_t^{(2)}$, but the double moving average cannot be calculated until 5 single moving averages have been calculated, that is, from time points 1 to 8. Using M_9 and $M_9^{(2)}$ to obtain the prediction after time 1 from time 9, $\hat{Y}_{9+1}$ is as follows:

$$\begin{aligned}\hat{Y}_{9+1} &= 2M_9 - M_9^{(2)} + 1\left(\frac{2}{5-1}\right)(M_9 - M_9^{(2)}) \\ &= 2\times 104.50 - 104.028 + 1\left(\frac{2}{5-1}\right)(104.50 - 104.028) = 105.2080\end{aligned}$$

The predicted values calculated in the same way are shown in the fifth column.

[Table 13.5.3] Double Moving Average of Consumer Price Index and One Time Forecast

Time t	Consumer Price Index Y_t	5-pt Single Moving Average M_t	5점 Double Moving Average $M_t^{(2)}$	One Time Forecast $\hat{Y}_{(t-1)+1}$
1	102.3			
2	102.8			
3	103.7			
4	104.1			
5	104.0	103.38		
6	104.3	103.78		
7	104.5	104.12		
8	104.9	104.36		
9	104.8	104.50	104.028	
10	104.8	104.66	104.284	105.2080
11	104.2	104.64	104.456	105.2240
12	104.4	104.62	104.556	104.9160
13	105.0	104.64	104.612	104.7160
14	105.5	104.78	104.668	104.6820
15	106.1	105.04	104.744	104.9480
16	106.7	105.54	104.924	105.4840
17	107.1	106.08	105.216	106.4640
18	107.0	106.48	105.584	107.3760
19	106.7	106.72	105.972	107.8240
20	107.4	106.98	106.36	107.8420
21	108.0	107.24	106.7	107.9100
22	107.7	107.36	106.956	108.0500
23	107.8	107.52	107.164	107.9660
24	108.3	107.84	107.388	108.0540

<Figure 13.5.4> Forecast using Double Moving Average of Consumer Price Index

B. Holt Double Exponential Smoothing Model

Holt proposed a model for a linear time series $Y_t = \beta_0 + \beta_1 \cdot t + \epsilon_t$ which uses a smoothing constant for the level and a smoothing constant for the trend. This is called **Holt's linear trend exponential smoothing model** or **two-parameters double exponential smoothing model**. Let $\hat{b}_0$ and $\hat{b}_1$ be the initial values of the intercept and slope, and α be the smoothing constant of the level and γ is the smoothing constant of the trend. The predicted values $\hat{Y}_t$, level $\hat{\beta}_0(t)$ and trend $\hat{\beta}_1(t)$ are as follows:

Predicted value: $\hat{Y}_t = \hat{\beta}_0(t-1) + \hat{\beta}_1(t-1),\ t = 1, 2, \cdots$

Level: $\hat{\beta}_0(t) = \alpha Y_t + (1-\alpha)\hat{Y}_t \quad t = 1, 2, \cdots$

Trend: $\hat{\beta}_1(t) = \gamma\{\hat{\beta}_0(t) - \hat{\beta}_0(t-1)\} + (1-\gamma)\hat{\beta}_1(t-1), \quad t = 1, 2, \cdots$

That is, the level is the weighted average of the current observed value Y_t and the predicted value $\hat{Y}_t$, and the trend is the weighted average of the level difference $\hat{\beta}_0(t) - \hat{\beta}_0(t-1)$between the time points t and $(t-1)$ and the trend $\hat{\beta}_1(t-1)$ at the time point (t-1). For this model, initial values of level $\hat{\beta}_0(0)$ and slope $\hat{\beta}_1(0)$ are required and the intercept and slope of the simple regression analysis of all observed values are widely used as initial estimates. Similar to the single exponential smoothing model, a value between 0.1 and 0.3 is often used to determine the smoothing constants α and γ.

The predicted values for the time point $T+\tau$ at time T using the trend exponential smoothing model are as follows:

$$\hat{Y}_{T+\tau} = \hat{Y}_T + \tau\hat{\beta}_1(T)$$

Such a trend exponential smoothing model is also a kind of heuristic method. That is, although logical, it is not based on any optimization such as least squares method.

The result of simple linear regression model to all data in [Table 13.5.4] is

as follows:

$$\hat{Y}_t = 102.574 + 0.2344\,t$$

[Table 13.5.4] is a calculation table for predicting the consumer price index with the Holt double exponential smoothing model using this initial values. The third column is the predicted value of the level $\hat{\beta}_0(t)$, the fourth column is the trend $\hat{\beta}_1(t)$, and the fifth column is the prediction $\hat{Y}_t = \hat{\beta}_0(t-1) + \hat{\beta}_1(t-1)$ obtained one time after each time point. Therefore, the forecast of the consumer price index for the next three months is as follows:

t = 25 : $\hat{Y}_{24+1} = \hat{Y}_{24} + 1 \times \hat{\beta}_1(24) = 108.19 + 0.237 = 108.42$
t = 26 : $\hat{Y}_{24+2} = \hat{Y}_{24} + 2 \times \hat{\beta}_1(24) = 108.19 + 2 \times 0.237 = 108.66$
t = 27 : $\hat{Y}_{24+3} = \hat{Y}_{24} + 3 \times \hat{\beta}_1(24) = 108.19 + 3 \times 0.237 = 108.90$

[

[Table 13.5.4] Forecasting using Holt Double Exponential Smoothing Model of Consumer Price Index

Time t	Consumer Price Index Y_t	Constant $\hat{\beta}_0(t)$	Trend $\hat{\beta}_1(t)$	One Time Forecast $\hat{Y}_{(t-1)+1}$
0		102.57	0.234	
1	102.3	102.76	0.229	102.81
2	102.8	102.97	0.227	102.99
3	103.7	103.25	0.232	103.20
4	104.1	103.54	0.239	103.48
5	104.0	103.80	0.241	103.78
6	104.3	104.07	0.243	104.04
7	104.5	104.33	0.245	104.31
8	104.9	104.61	0.249	104.58
9	104.8	104.85	0.248	104.86
10	104.8	105.07	0.245	105.10
11	104.2	105.20	0.234	105.31
12	104.4	105.33	0.224	105.44
13	105.0	105.50	0.218	105.56
14	105.5	105.70	0.216	105.72
15	106.1	105.93	0.218	105.91
16	106.7	106.20	0.223	106.15
17	107.1	106.49	0.230	106.43
18	107.0	106.75	0.233	106.72
19	106.7	106.96	0.230	106.98
20	107.4	107.21	0.232	107.19
21	108.0	107.50	0.238	107.44
22	107.7	107.73	0.237	107.73
23	107.8	107.95	0.236	107.97
24	108.3	108.20	0.237	108.19

<Figure 13.5.5> shows the predicted values using the Holt's double exponential smoothing model.

<Figure 13.5.5> Forecasting using Holt Double Exponential Smoothing Model of CPI

13.6 Seasonal Model and Forecasting

- As a seasonal time series model, a multiplicative model using a central moving average and a Holt-Winters model are introduced.

13.6.1 Seasonal Multiplicative Model

- Assume that a time series Y_t with a seasonal period L can be expressed as the product of a trend (T), a seasonal (S), and an irregular component (I) as follows:

$$Y_t = T_t \cdot S_t \cdot I_t .$$

- **The ratio to moving average method** removes the trend and irregular components to obtain the seasonal component as follows

(Step 1) For the time series, find the L-point centered moving average. This moving average represents the trend component T_t after removing seasonal component and irregular component from the time series.

(Step 2) Divide the time series Y_t by the trend component T_t obtained in Step 1. This value implies the seasonal component and the irregular component $S_t \cdot I_t$, and is called the seasonal ratio.

$$\frac{Y_t}{T_t} = S_t \cdot I_t$$

(Step 3) Calculate the trimmed average for each seasonal ratio obtained in Step 2. This is the seasonal index, but the normalization should be performed so that the sum of the seasonal indices is L.

- After obtaining the seasonal index as shown, dividing the original time series data by the seasonal index. The results is called a **deseasonal time series.**

Deseasonal time series: $D_t = \dfrac{Y_t}{S_t} = TI$

- This deseasonal time series D_t implies TI. An appropriate time series model is applied to this deseasonal data and predict the future vaules. Then multiply the corresponding seasonal index to obtain the final predicted value of the desired season.
- [Table 13.6.1] shows a company's quarterly sales. Since the seasonal period is 4, the 4-point centered moving average is as shown in column 4 of the table. By dividing the original time series by a 4-point centered moving average, the seasonal ratio in column 5 can be calculated.

$$\frac{Y_t}{T_t} = S_t \cdot I_t$$

[Table 13.6.1] Quarterly Sales of a Company

	① Year Quarter	② Sales Y_t	③ 4-point MA	④ Centered 4-point MA	⑤ Seasonal Ratio	⑥ Deseasonal Data D_t
1	2018 Quarter 1	75				62.552
2	Quarter 2	60				65.502
3	Quarter 3	54	62.000	63.375	0.852	63.754
4	Quarter 4	59	64.750	65.375	0.902	56.840
5	2019 Quarter 1	86	66.000	67.125	1.281	71.726
6	Quarter 2	65	68.250	70.875	0.917	70.961
7	Quarter 3	63	73.500	74.000	0.851	74.380
8	Quarter 4	80	74.500	75.375	1.061	77.071
9	2020 Quarter 1	90	76.250	76.625	1.175	75.063
10	Quarter 2	72	77.000	77.625	0.928	78.603
11	Quarter 3	66	78.250	79.500	0.830	77.922
12	Quarter 4	85	80.750	81.500	1.043	81.888
13	2021 Quarter 1	100	82.250	83.000	1.205	83.403
14	Quarter 2	78	83.750	84.750	0.920	85.153
15	Quarter 3	72	85.750			85.006
16	Quarter 4	93				89.595

- [Table 13.6.2] shows the seasonal ratio by year and quarter. If the maximum and minimum values are removed for each quarter and the average is obtained (trimmed average), it is the seasonal index in column 6. Since the sum of these values is 4.0197, the seasonal index in column is normalized as in column 7.

$\hat{S}_1$ = 1.199, $\hat{S}_2$ = 0.916, $\hat{S}_3$ = 0.847, $\hat{S}_4$ = 1.038

[Table 13.6.2] Seasonal Index

① Year / Quarter	② 2018	③ 2019	④ 2020	⑤ 2021	⑥ Trimmed Mean of Seasonal Ratio	⑦ Seasonal Index S_t
1st Quarter		1.281	1.175	1.205	1.205	1.199
2nd Quarter		0.917	0.928	0.920	0.920	0.916
3rd Quarter	0.852	0.851	0.830		0.851	0.847
4th Quarter	0.902	1.061	1.043		1.043	1.038
					sum 4.019	

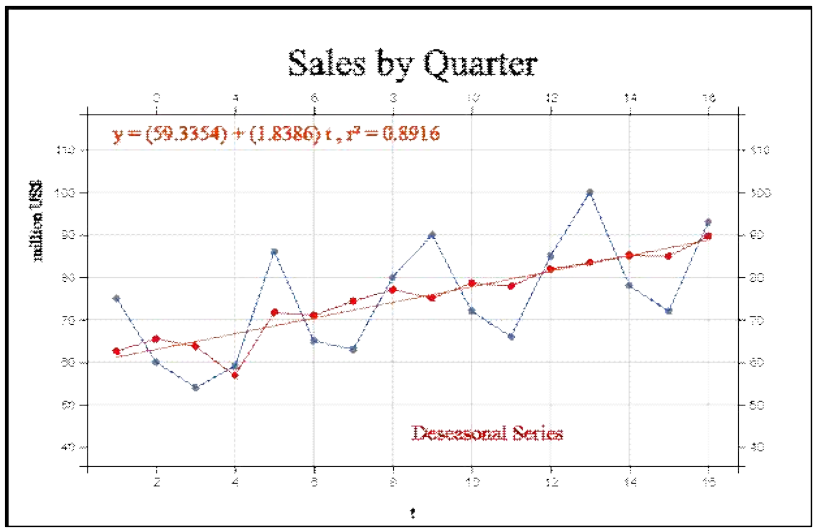
Column 6 of [Table 13.6.1] shows the non-seasonal data D_t obtained by dividing the original data by the seasonal index of each quarter. The linear regression line for this non-seasonal data is as follows (<Figure 13.6.1>).,

$$D_t = 59.335 + 1.839\,t$$

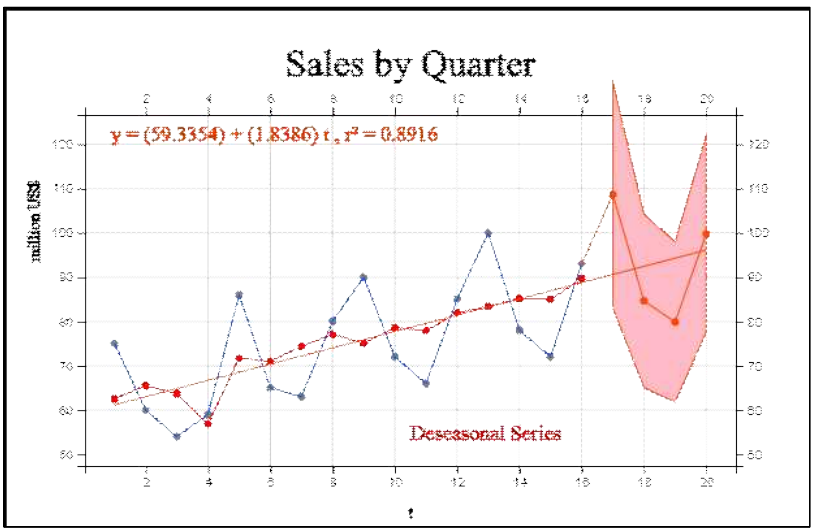
Therefore, the forecast for the next one year is as follows.

$$\text{Time 17}: \hat{Y}_{17} = (59.335 + 1.839 \times 17) \times 1.199 = 108.627$$
$$\text{Time 18}: \hat{Y}_{18} = (59.335 + 1.839 \times 18) \times 0.916 = 84.672$$
$$\text{Time 19}: \hat{Y}_{19} = (59.335 + 1.839 \times 19) \times 0.847 = 79.852$$
$$\text{Time 20}: \hat{Y}_{20} = (59.335 + 1.839 \times 20) \times 1.038 = 99.767$$

<Figure 13.6.2> is a graph of seasonal forecasts.



<Figure 13.6.1> Linear Forcasting Model of Deseasonal Sale



<Figure 13.6.2> 1 Year Forcasting of Quarterly Sales

13.6.2 Holt-Winters Model

- Assume that a time series with a seasonal period L is observed over m cycles as follows:

	Season 1	Season 2	$\cdots$	Season L
Cycle 1	Y_1	Y_2	$\cdots$	Y_L
Cycle 2	Y_{L+1}	Y_{L+2}	$\cdots$	Y_{2L}
	$\cdots$	$\cdots$		$\cdots$
Cycle m	$Y_{(m-1)L+1}$	$Y_{(m-1)L+2}$	$\cdots$	Y_{mL}

- The Holt-Winters model is an extension of Holt's linear double exponential smoothing method studied in the previous section to a seasonal model. It consists of a level component l_t, a trend component b_t, and a seasonal component s_t. There are additive model and multiplicative model, but only the multiplication model is introduced here.

$$\hat{Y}_{t+h} = (l_t + h\,b_t) \cdot s_{t+h-m(k+1)}$$
$$l_t = \alpha \frac{y_t}{s_{t-m}} + (1-\alpha)(l_{t-1} + b_{t-1})$$
$$b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$$
$$s_t = \gamma \frac{y_t}{l_{t-1} + b_{t-1}} + (1-\gamma)s_{t-m}$$

Here k is integer part of $(h-1)/m$

l_t is a time series level, which means the exponential smoothing of the current level ($\frac{y_t}{s_{t-m}}$) with seasonality removed and the value predicted one time ago $(l_{t-1} + b_{t-1})$. b_t is the slope which is the exponential smoothing of the slope of the current time point $(l_t - l_{t-1})$ and the previous time point (b_{t-1}). s_t is a seasonal index which is the exponential smoothing of the current seasonal

component ($\frac{y_t}{l_{t-1}+b_{t-1}}$) and the seasonal component of the previous season s_{t-m}.

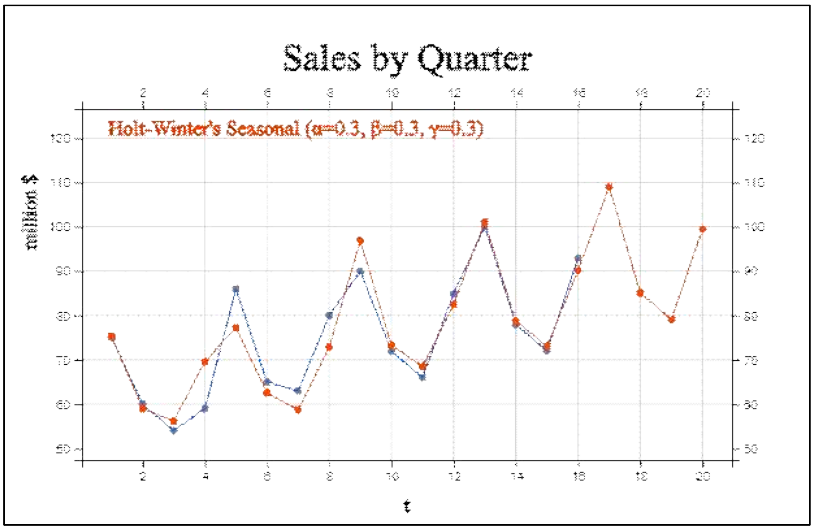
- [Table 13.6.3] calculates exponential smoothing values of level, slope, and seasonal indices using the Holt-Winters model with α = 0.3, β = 0.3, γ = 0.3 to the quarterly sales of a company. The last column is one time prediction at time $(t-1)$, $\hat{Y}_{(t-1)+1}$. The initial values $l_0$and b_0 are the intercept and slope of the linear regression model for all data, and the initial values of the seasonal index are calculated by the model $Y = T \times S \times I$.

[Table 13.6.3] Holt-Winters Forecasting Model of Quarterly Sales

	Year Quarter	Sales Y_t	Level l_t	Slope b_t	Seasonal s_t	One Time Forecast $\hat{Y}_{(t-1)+1}$
-3					1.1991	
-2					0.9159	
-1					0.8472	
0			61.2	1.61	1.0378	
1	2018 Quarter 1	75	62.7312	1.5863	1.1976	75.315
2	Quarter 2	60	64.6753	1.6937	0.9210	58.908
3	Quarter 3	54	65.5795	1.4568	0.8371	56.230
4	Quarter 4	59	63.9809	0.5402	0.9905	69.570
5	2019 Quarter 1	86	66.7081	1.1963	1.2382	77.270
6	Quarter 2	65	68.7061	1.4368	0.9319	62.539
7	Quarter 3	63	71.6766	1.8969	0.8555	58.720
8	Quarter 4	80	75.7320	2.5445	1.0195	72.874
9	2020 Quarter 1	90	76.5997	2.0414	1.2117	96.920
10	Quarter 2	72	78.2283	1.9176	0.9270	73.282
11	Quarter 3	66	79.2477	1.6481	0.8459	68.561
12	Quarter 4	85	81.6382	1.8708	1.0289	82.477
13	2021 Quarter 1	100	83.2158	1.7829	1.2074	101.184
14	Quarter 2	78	84.7427	1.7061	0.9242	78.791
15	Quarter 3	72	86.0501	1.5865	0.8420	73.124
16	Quarter 4	93	88.4618	1.8341	1.0386	90.169

- <Figure 13.6.3> is the Holt-Winters forecast for the next one year and is calculated as follows:

$$\hat{Y}_{16+1} = (l_{16}+1\times b_{16})\cdot s_{13} = (88.4618+1.8341)\times 1.2074 = 109.024$$
$$\hat{Y}_{16+2} = (l_{16}+2\times b_{16})\cdot s_{14} = (88.4618+2\times 1.8341)\times 0.9242 = 85.144$$
$$\hat{Y}_{16+3} = (l_{16}+3\times b_{16})\cdot s_{15} = (88.4618+3\times 1.8341)\times 0.9420 = 79.115$$
$$\hat{Y}_{16+4} = (l_{16}+4\times b_{16})\cdot s_{16} = (88.4618+4\times 1.8341)\times 1.0386 = 99.495$$



<Figure 13.6.3> Holt-Winters Forecasting of Quarterly Sales

Exercise

For the next exercise (13.1 – 13.4), draw graph of the time series data, Apply an appropriate smoothing method and transformation, find an appropriate prediction model to predict the next year.

13.1 The following table provides data on the number of items damaged in shipment during 2001 - 2014 for a manufacturer.

Year	Items	Year	Items
2001	533	2008	291
2002	373	2009	228
2003	132	2010	204
2004	555	2011	349
2005	168	2012	234
2006	281	2013	209
2007	175	2014	176

13.2 The following table shows the sales volume (in thousands of dollars) of a retail store between 2001-2014.

Year	Sales	Year	Sales
2001	815	2008	12,529
2002	1,276	2009	12,824
2003	4,752	2010	13,777
2004	7,535	2011	15,379
2005	10,122	2012	18,705
2006	9,642	2013	17,632
2007	14,100	2014	16,571

13.3 The following table shows the number of items repaired during a company's warranty period between 2001 and 2014.

Year	Items	Year	Items
2001	749	2008	611
2002	709	2009	600
2003	700	2010	574
2004	678	2011	559
2005	611	2012	543
2006	641	2013	534
2007	631	2014	524

13.4 The following table shows the annual sales (unit: billion $) of a company for 11 years.

Year	Sales	Year	Sales
2012	12	2018	20
2013	14	2019	22
2014	18	2020	27
2015	20	2021	24
2016	18	2022	30
2017	16		

13.5 The following data show the price of silver and crude oil between 2000 and 2015, respectively. Find the percentage change of silver and crude oil and the price indices and overlay them on one picture.

Year	Silver ($ /ounce)	Crude Oil ($ /barrel)	Year	Silver	Crude Oil
2000	1.771	1.80	2008	5.440	15.40
2001	1.546	2.18	2009	11.090	18.00
2002	1.684	2.48	2010	20.633	28.00
2003	2.558	5.18	2011	10.481	32.00
2004	4.708	10.46	2012	7.950	34.00
2005	4.419	11.51	2013	11.439	30.00
2006	4.353	11.51	2014	8.141	26.00
2007	4.620	12.70	2015	6.192	26.00

13.6 The following table shows the number of skis sold by a sports merchandise seller in 2017-2021.

1) Predict the next year with a multiplicative seasonal model.
2) Predict the next year using the Holt-Winters seasonal model.

Year	1	2	3	4	5	6	7	8	9	10	11	12
2017	0	2	10	4	89	33	11	4	17	5	17	0
2018	3	0	5	4	14	23	7	11	11	4	4	8
2019	9	2	46	11	14	30	22	4	7	4	0	2
2021	13	4	56	30	90	20	15	11	6	5	1	7
2021	4	12	6	10	17	32	24	9	10	5	17	1